Minimized Input-Energy Gain based Static Decoupling Control for Linear Over-actuated Systems with Sinusoidal References

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***Abstract*— This paper introduces a convex optimization procedure to assign the additional degrees of freedom (DoF) in decoupling control of *over-actuated* linear systems via static linear feedback. By a suitable choice of the DoFs, the tasks of system stabilization and optimization can be separated. This involves the parametrization of steady state trajectories of the system at the same time. Exploiting the resulting framework, it is feasible to formulate a linear semidefinite program which accounts for minimization of the input-energy gain regarding sinusoidal references with known frequencies. Representing the latter by an exogenous system, an additional constraint guarantees a performance increase for arbitrary exogenous initial values. The effectiveness of the approach is shown in simulations and compared to previous results.**

I. Introduction

Many practical applications involve *over-actuation* as an inherent system property due to safety reasons. In case of actuator faults, this allows control reconfiguration or fault-tolerant control [1]. Typical examples are aircraft, automobiles or chemical processes, e.g., [2], [3], [4].

During normal operation, i.e., healthy system conditions the question arises how to exploit the surplus of actuators in a beneficial way. In order to answer this question properly, two types of *over-actuation* have to be distinguished.

First, consider a linear system with an input matrix having linearly dependent columns. If it is possible to substitute the control inputs by virtual control inputs (VCI) leading to a square system, i.e., the number of outputs equals the number of VCIs we arrive at the task of *control allocation*. This allows a modular controller structure with a control law for VCIs derived by any favored design for square systems and an allocation of original inputs, see survey [1].

Second, the input matrix exhibits more linearly independent columns than outputs. To the best of our knowledge, a generalized controller structure does not exist. Instead, usage of the additional inputs is often incorporated in the chosen design procedure. For instance, the influence of input redundancy in the infinite standard regulator problem is analyzed in [5] and noticeable progress in output regulation based control took place, e.g., [6].
In many practical applications the decoupling of the input-output path is desired, which will be the focus of this paper. Regarding static feedback, the results reach from approaches for nonlinear systems based on input-output linearization [4] to LMI-based control design for constrained linear systems stabilizing internal dynamics [7]. These have in common that actuator costs are minimized regarding the current point in time only. Thus, the influence on system dynamics of the DoFs is neglected in this regard, which is an important aspect [8]. In the linear case, many publications deal with the yet unsolved and controversially discussed problem of finding sufficient and necessary conditions, e.g., [9].

In [10], the authors adopted the methodology of robust control by minimizing the H_∞-norm of the transfer function from reference signals to control inputs, which is equal to the input-energy gain. The implementation by application of the *bounded real lemma* (BRL) is promising since the influence of the DoFs on internal dynamics is regarded. Nonetheless, it is impossible to incorporate any information on the references.

This motivates us to extend the results of [10] by explicitly involving provided information on reference signals in the optimization procedure. We regard sinusoidal references with known frequencies but arbitrary amplitudes and phases at stage of control design. This allows for example to approximate signals by finite Fourier decomposition. In spite of increasing complexity due to avoidance of the BRL, we will end up with a linear semidefinite program (SDP) accounting for all goals, therefore, yielding a static decoupling control based on minimal input-energy gain in steady state (ss).

Hence, the next section defines the system structure and reference signals as well as provides a problem statement. In Section III, the methodology for decoupling static feedback design is presented. Then in Section IV the DoFs are parametrized and the linear SDP is derived. Finally, Section V indicates the effectiveness of the approach compared to previous results.

Mathematical notations: The zero matrix 0 has appropriate dimensions if not denoted explicitly by $0_{a,b} \in \mathbb{R}^{a\times b}$. It follows analogously the identity matrix I with appropriate dimensions, $I_a \in \mathbb{R}^{a\times a}$ respectively. For arbitrary matrices M_i we write

$$\mathrm{diag}(M_1, M_2) = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

and $\mathrm{diag}(M_1, M_2, M_3, \ldots) = \mathrm{diag}\left(M_1, \mathrm{diag}(M_2, M_3, \ldots)\right)$ iteratively. The basis of the null space of matrix M is denoted by $M^\perp$ whereas M^+ is the *Moore-Penrose* pseudoinverse. With $M \prec 0$, $M \preceq 0$ the symmetric matrix M is negative definite, negative semidefinite respectively and the spectrum is given by $\sigma(M)$. An appropriate dimensional vector with i-th element equals 1 else 0 is represented by $e_i \in \mathbb{R}^a$. The subscript ss accounts for steady state evolution of a time-varying variable $x_{\mathrm{ss}}(\cdot)$.

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II. PRELIMINARIES

A. System Setup and Exogenous Inputs

Let the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ as well as $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ be given. Please note, neglecting a feedthrough matrix is not a limitation, since all subsequent steps can be easily adopted for systems with feedthrough matrix. It further holds $m > p$, i.e., system (1) is *over-actuated* and

$$p < \text{rank } B \leq m \tag{2}$$

is satisfied. This case is often referred to as *weak input redundancy* [6] based on [11].

In the sequence, only sinusoidal reference trajectories $w_i(\cdot)$ for each output y_i of form

$$w_i(\cdot) = w_{i,C} + \sum_{v=1}^{n_{i,w}} w_{i,v} \cos(\omega_{i,v} t + \alpha_{i,v}). \tag{3}$$

with known frequencies $\omega_{i,v}$ are regarded. This allows to decompose or approximate signals by finite Fourier series. References (3) can be given by an observable exo-system

$$\begin{aligned} \dot{x}_{\text{exo}} &= S x_{\text{exo}}, \\ w &= Q x_{\text{exo}} \end{aligned} \tag{4}$$

with $w \in \mathbb{R}^p$, $S \in \mathbb{R}^{n_{\text{exo}}\times n_{\text{exo}}}$, $Q \in \mathbb{R}^{p\times n_{\text{exo}}}$ and yet unspecified initial values $x_{\text{exo},0} = x_{\text{exo}}(0) \in \mathbb{R}^{n_{\text{exo}}}$. The latter guarantees that the time series (3) does not need to be known beforehand. At stage of control design the only information assumed provided are the frequency spectra $\Omega_i = \{\omega_{i,v},\, v = 1,\dots,n_{i,\omega}\}$ of signal w_i.

At this point, the following formulation of the exogenous system might seem rather technical but will be proven helpful in Section IV-B. Matrix S is defined block diagonal

$$S = \text{diag}(S_C, S_1, ..., S_{N_\Omega}) \tag{5}$$

wherein

$$S_C = 0_{n_C\times n_C} \tag{6a}$$

$$S_v = \text{diag}\left(\begin{bmatrix} 0 & \omega_v \\ -\omega_v & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega_v \\ -\omega_v & 0 \end{bmatrix}, \dots \right) \tag{6b}$$

for $v = 1,\dots,N_\Omega$ with cardinality N_Ω of $\Omega = \Omega_1 \cup \dots \cup \Omega_p \setminus \{0\}$. Furthermore, $S_v \in \mathbb{R}^{2N_{\omega,v}\times 2N_{\omega,v}}$ with ω_v being element of the spectra Ω_i of $N_{\omega,v} \leq p$ references w_i. The associated state vectors of related subsystems are $x_{\text{exo},C} \in \mathbb{R}^{n_C}$ and $x_{\text{exo},v} \in \mathbb{R}^{2N_{\omega,v}}$. It follows $n_{\text{exo}} = n_C + \sum_{v=1}^{N_\Omega} 2N_{\omega,v}$. With (6b), it is respected that a signal of frequency ω_v can occur in several w_i with different amplitudes, phases and requires an own exo-system each.

Moreover, assuming $\omega_i/\omega_j \in \mathbb{Q}\ \forall \omega_i, \omega_j \in \Omega$ is satisfied, which is always approximately fulfilled with arbitrary high accuracy, a constant $k_v \in \mathbb{N}$ exists for each angular frequency $\omega_v = 2\pi/T_v \in \Omega$ such that

$$T = k_v T_v\ \forall v, \tag{7}$$

which is defined the period length of w in (4). Then, the following lemma can be shown straightforwardly

Lemma 1: If $\frac{\omega_i}{\omega_j} \in \mathbb{Q}$, i.e., $\exists\ T = k_i \frac{2\pi}{\omega_i} = k_j \frac{2\pi}{\omega_j}$ with $k_i, k_j \in \mathbb{N}$ and $t_0 \geq 0$ then

1) $\int_{t_0}^{t_0+T} \cos(\omega_i t + \alpha_i) dt = 0$
2) for $\omega_i \neq \omega_j$: $\int_{t_0}^{t_0+T} \cos(\omega_i t + \alpha_i) \cos(\omega_j t + \alpha_j) dt = 0$
3) with $l \in \mathbb{Z}$: $\int_{t_0}^{t_0+T} \cos(\omega_i t + \alpha_i) \cos(\omega_i t + \alpha_i + \frac{\pi}{2}(2l + 1)) dt = 0$
4) $\int_{t_0}^{t_0+T} \cos^2(\omega_i t + \alpha_i) dt = \frac{T}{2}$

hold $\forall \alpha_i, \alpha_j \in \mathbb{R}$.

B. Problem Statement

The goal of this contribution is to find a static control law for system (1) that guarantees

I) decoupling of the exogenous inputs w to system outputs y such that the transfer function $G_{y,w}(s)$ is diagonal,
II) if desired, asymptotic tracking of the time-varying references $w(\cdot)$ and
III) minimization of the input-energy gain γ, which is generally determined by $\int_0^\infty u^\top u dt < \gamma^2 \int_0^\infty w^\top w dt$ as in [10], regarding the steady state evolution x_{ss} induced by the reference inputs w.

The next section addresses I), II) and introduces the additional degrees of freedom (DoFs). Based on a suitable parametrization, a convex optimization problem is formulated to account for III) in Section IV.

III. DECOUPLING FOR OVER-ACTUATED LINEAR SYSTEMS

A static decoupling controller for system (1) can be deduced from the method of Falb-Wolovich [12]. With $i = 1,\dots,p$ the derivatives of the scalar i-th output $y_i = c_i^\top x$ are

$$\begin{aligned} \dot{y}_i &= c_i^\top A x, \\ &\vdots \\ y_i^{(\delta_i - 1)} &= c_i^\top A^{\delta_i - 1} x, \\ y_i^{(\delta_i)} &= c_i^\top A^{\delta_i} x + \underbrace{c_i^\top A^{\delta_i - 1} B}_{\neq 0} u, \end{aligned}$$

if

$$c_i^\top A^j B = 0\ \forall\ 0 \leq j < \delta_i - 1$$

holds. Therefore, the relative degree δ_i of output y_i of system (1) is defined as in the case of square systems by

$$\delta_i = \min_{j\in\mathbb{N}^+} \{j : c_i^\top A^{j-1} B \neq 0\}.$$

Requiring

$$\underbrace{\begin{bmatrix} c_1^\top A^{\delta_1 - 1} B \\ \vdots \\ c_p^\top A^{\delta_p - 1} B \end{bmatrix}}_{D} u = \begin{bmatrix} -c_1^\top A^{\delta_1} \\ \vdots \\ -c_p^\top A^{\delta_p} \end{bmatrix} x + \nu \tag{8}$$

with non-square decoupling matrix $D \in \mathbb{R}^{p \times m}$ and the virtual input $\nu \in \mathbb{R}^p$ yields a chain of integrators for every output

$$y^{(\delta_i)} = \nu_i. \tag{9}$$

With the relative degree $\delta = \sum_{i=1}^{p} \delta_i \leq n$ of system (1) the closed loop consists of p integrator chains (9) of order δ_i called external dynamics and on the opposite the internal dynamics of order $n - \delta$ with state vector $\eta \in \mathbb{R}^{n-\delta}$.

If asymptotic tracking behavior of time-varying references $w_i(\cdot) \in \mathbb{R}$ is desired, then choosing

$$\nu_i = w_i^{(\delta_i)} - \sum_{j=0}^{\delta_i - 1} a_{i,j}(y_i^{(j)} - w_i^{(j)}) \tag{10}$$

gives the dynamics

$$\tilde{y}_i^{(\delta_i)} + \sum_{j=0}^{\delta_i - 1} a_{i,j} \tilde{y}_i^{(j)} = 0 \tag{11}$$

of output tracking error $\tilde{y}_i = y_i - w_i$ which converges asymptotically to zero based on an appropriate choice of coefficients $a_{i,j}$. Evidently, the time-varying reference $w_i(\cdot)$ being δ_i-times differentiable is necessary, which is satisfied by any (3). If asymptotic tracking is not required, all derivatives of $w_i^{(j)}$ with $j \geq 1$ can be neglected in (10) and in all subsequent equations.

The virtual input ν is given by

$$\nu = -\begin{bmatrix} \sum_{j=0}^{\delta_1 - 1} a_{1,j} c_1^\top A^j \\ \vdots \\ \sum_{j=0}^{\delta_p - 1} a_{p,j} c_p^\top A^j \end{bmatrix} x + \begin{bmatrix} w_1^{(\delta_1)} + \sum_{j=0}^{\delta_1 - 1} a_{1,j} w_1^{(j)} \\ \vdots \\ w_p^{(\delta_p)} + \sum_{j=0}^{\delta_p - 1} a_{p,j} w_p^{(j)} \end{bmatrix}. \tag{12}$$

Now, regarding $w_i^{(j)} = q_i^\top S^j x_{\text{exo}}$ based on (4) yields

$$N x_{\text{exo}} = \begin{bmatrix} q_1^\top \left(S^{\delta_1} + \sum_{j=0}^{\delta_1 - 1} a_{1,j} S^j \right) \\ \vdots \\ q_p^\top \left(S^{\delta_p} + \sum_{j=0}^{\delta_p - 1} a_{p,j} S^j \right) \end{bmatrix} x_{\text{exo}}$$

equivalent to the last term in (12). Substituting ν in (8) and introducing a suitable matrix M gives

$$Du = -Mx + N x_{\text{exo}}. \tag{13}$$

With the decoupling matrix D being non-square, the following assumption guarantees the existence of a u satisfying (13), cf. [10].

Assumption 1: It holds rank $D = p$.

With Ass. 1, the right inverse $D^+ = D^\top \left(DD^\top\right)^{-1}$ with $DD^+ = I_{p,p}$ exists. Exploiting the Rank-Nullity-Theorem, there also exists a $m - p$ dimensional null space null$(D) =$ span$\left(D^\perp\right)$. Wherein $D^\perp \in \mathbb{R}^{m \times (m-p)}$ with rank $D^\perp = m - p$ and $DD^\perp = 0_{p,m-p}$.

Furthermore, the next assumption ensures the existence of a control law that guarantees stable internal dynamics.

Assumption 2: The pair $\left(A - BK, BD^\perp\right)$ with K satisfying $DK = M$ is stabilizable.

Following the discussion in Section I, the given assumptions 1 and 2 are not necessary but sufficient for the existence of a static control law decoupling (1).

With the provided information, it is possible to setup the following theorem which shares characteristics with [10].

Theorem 1: Let system (1) with relative degrees δ_i be *over-actuated*, rank $B > p$ and Ass. 1 as well as Ass. 2 are satisfied. With the decoupling matrix D, matrices M and N as in (13), $D^\perp \in \mathbb{R}^{m \times (m-p)}$ satisfying $DD^\perp = 0_{p,m-p}$, $D^+ \in \mathbb{R}^{m \times p}$ satisfying $DD^+ = I_{p,p}$ and choice of $a_{i,j}$ with $i = 1, \ldots, p$ and $j = 0, \ldots, \delta_i - 1$ such that (11) is asymptotically stable $\forall i$, the control law

$$u = -Kx + F x_{\text{exo}} + D^\perp z \tag{14}$$

with

$$K = D^+ M,$$
$$F = D^+ N$$

guarantees decoupling of the system outputs y as well as asymptotic tracking of the time-varying reference input $w(\cdot)$ given by the autonomous system (4). Furthermore, the new input $z \in \mathbb{R}^{m-p}$ allows for stabilization of the pair $\left(A - BK, BD^\perp\right)$ and therefore stability of the closed loop.

Proof: can be deduced form the discussion up to Theorem 1 in a straightforward manner. ■

As can be seen from (14), the remaining DoFs in the design process are cascaded by the new input z similar as in [13]. Consequently, it is left to show how z should be chosen in order to stabilize the closed loop and to account for minimized input-energy gain γ at the same time. For this purpose, a suitable parametrization of z will be introduced in the next section and an adequate optimization problem will be formulated to account for all design goals.

IV. MINIMIZATION OF INPUT-ENERGY GAIN FOR SINUSOIDAL REFERENCES

In this section, part A parametrizes the DoFs and formulates an optimization problem regarding input-energy gain. In part B, we derive the corresponding linear SDP and introduce an additional upper bound via constraints.

Therefore, it is assumed that a control law as in Theorem 1 can be implemented providing the DoFs z and the exogenous inputs are defined as in Section II-A.

A. Parametrization of DoFs and formulation of optimization problem

Following [14], in case of a system with periodic inputs w and periodic outputs u with period length T the input-energy gain γ satisfies

$$\int_{t_0}^{t_0+T} u^\top u dt < \gamma^2 \int_{t_0}^{t_0+T} w^\top w dt \tag{15}$$

with $t_0 \geq 0$, which is also equal to the root mean square gain of the system. Assuming stability of the closed loop, periodicity will account for the steady state trajectory of system (1) with control law (14) regarding periodic inputs (3) for $t_0 \gg 0$.

Thus, an optimization problem regarding the input-energy gain can be stated as

$$\begin{aligned}&\min_{\gamma>0,z(\cdot)} \gamma\\ &\text{subject to:}\\ &\quad (15), (14), (1) \text{ and } (4) \text{ with}\\ &\quad x(0)=x_0 \text{ and } x_{\text{exo}}(0)=x_{\text{exo},0} \text{ for } t_0 \gg 0\end{aligned} \tag{16}$$

with $z(\cdot):\mathbb{R}\to\mathbb{R}^{m-p}$. Considering the structure of (14) and following [10], a parametrization $z=f(x,x_{\text{exo}})$ is reasonable. In order to achieve a static linear feedback law and to allow separation of the stabilization and optimization tasks, $z(\cdot)$ is parametrized as

$$z = Z_K\tilde{x} + Z_F x_{\text{exo}} \tag{17}$$

with $Z_K \in \mathbb{R}^{m-p\times n}$, $Z_F \in \mathbb{R}^{m-p\times n_{\text{exo}}}$ and

$$\tilde{x} = x - \Pi x_{\text{exo}} \tag{18}$$

with steady state evolution $x_{\text{ss}} = \Pi x_{\text{exo}}$ of system (1). Hence, $\Pi \in \mathbb{R}^{n\times n_{\text{exo}}}$ satisfies the Sylvester equation

$$\Pi S = (A-BK)\Pi + BF + BD^{\perp}Z_F; \tag{19}$$

the methodology is adopted from [15], thus exhibiting similarities compared to [6]. Equation (18) is a transformation of the system state x, which applied to the dynamics of (1) yields under consideration of (19)

$$\dot{\tilde{x}} = (A-BK+BD^{\perp}Z_K)\tilde{x}. \tag{20}$$

Because of Ass. 2, the autonomous system (20) can be stabilized by proper choice of Z_K, which also guarantees asymptotic stability of the closed loop. Since $\tilde{x}\to 0$ asymptotically, it can be assumed $z \to Z_F x_{\text{exo}}$ and $x \to \Pi x_{\text{exo}}$ for $t_0 \gg 0$. Thus, the optimization of input-energy gain γ in steady state depends on Z_F, Π only. Concluding, a separation of the two tasks first stabilization via Z_K, and second optimization via Z_F, Π is accomplished. Furthermore, in case of asymptotic tracking injected by (K,F), respecting $\tilde{x}\to 0$ gives $\widetilde{y} = (C\Pi - Q)\,x_{\text{exo}} \to 0$; thus, $C\Pi = Q$ always holds.

With all this in mind, the optimization problem can be reformulated for the steady state evolution assuming $\tilde{x}\to 0$ for $t_0 \gg 0$

$$\min_{\gamma>0,Z_F,\Pi} \gamma \tag{21a}$$

subject to:

$$\int_{t_0}^{t_0+T} u_{\text{ss}}^{\top}u_{\text{ss}}dt < \gamma^2 \int_{t_0}^{t_0+T} w^{\top}w dt \text{ with} \tag{21b}$$

$$u_{\text{ss}} = (-K\Pi + F + D^{\perp}Z_F)x_{\text{exo}}, \tag{21c}$$

$$\Pi S = (A-BK)\Pi + BF + BD^{\perp}Z_F \text{ and} \tag{21d}$$

$$\text{exo-dynamics (4) } \forall x_{\text{exo}}(t_0) = x_{\text{exo},0} \in \mathbb{R}^{n_{\text{exo}}} \tag{21e}$$

with the arguments of the optimum given by $Z_{F,\text{opt}}$, Π_{opt} and γ_{opt}. In this connection, $\Pi_{\text{opt}}x_{\text{exo}}$ can be referred as a parametrization of the system steady state trajectories resulting in minimal input-energy gain. Then $\tilde{x}$ describes the error during transient behavior. Since $DD^{\perp}=0$ and, consequently, z not influencing external dynamics, $\Pi_{\text{opt}}x_{\text{exo}}$ particularly parametrizes the trajectories of the internal dynamics which are shaped by the feed forward $Z_{F,\text{opt}}$ through $BD^{\perp}$ regarding (19). Furthermore, the solution of (21) applies $\forall x_{\text{exo},0}$ such that at stage of control design the amplitudes and phases of (3) are not restricted yet.

Remark 1: In [10] the parametrization

$$z = Z_K^{*}x + Z_F^{*}x_{\text{exo}} \tag{22}$$

is chosen, which leads to $z = Z_K^{*}\tilde{x} + (Z_K^{*}\Pi + Z_F^{*})x_{\text{exo}}$ using (18). This seems to give Z_K^{*} room to influence

$$\begin{aligned}\Pi S &= (A-BK+BD^{\perp}Z_K^{*})\Pi + BF + BD^{\perp}Z_F^{*},\\ u_{\text{ss}} &= (-K\Pi + F + D^{\perp}(\Pi Z_K^{*} + Z_F^{*}))x_{\text{exo}}.\end{aligned} \tag{23}$$

Nonetheless, it is always possible to substitute $Z_K = Z_K^{*}$, $Z_F = -Z_K^{*}\Pi + Z_F^{*}$, which again gives (17) and (21c).
With argumentation in Remark 1, there is no point in including Z_K into optimization problem (21) as the system trajectories $\Pi_{\text{opt}}\,x_{\text{exo}}$ can be reached solely using a prefilter Z_F and additionally (23) would involve undesired bilinear terms ΠZ_K^{*}. Hence, the separation approach provided here is beneficial, e.g., Z_K can be freely chosen to meet design goals like decay rate of $\tilde{x}$ and a convex formulation of the optimization problem is feasible.

Remark 2: Based on [14], it is well known that (15) is equal to

$$\|G_{u,w}(j\omega)\|_{\infty} < \gamma \tag{24}$$

with references w assumed to be generally T-periodic with unspecific T and transfer function $G_{u,w}(s)$ of the system with input w and output u. Applying the *bounded real lemma* [14] leads to an LMI equivalent to (24) which can be used to formulate a linear SDP. This method was adopted in [10] in order to find optimized Z_K^{*} and Z_F^{*} in (22).
In the next section it is shown that (21) is convex by formulating a linear SDP.

B. Linear SDP accounting for minimum input-energy gain in case of sinusoidal references

The formulation of a linear SDP is beneficial as it preserves convexity and allows to use a wide variety of solvers and interfaces for numerical implementation.

In order to derive a linear SDP from (21), the next lemma gives an alternative form of the integral terms in (21b). Therein, the block diagonal structure of S in (6) is exploited by help of the partition $\Pi = \begin{bmatrix}\Pi_C & \dots & \Pi_v & \dots\end{bmatrix}$, $F = \begin{bmatrix}F_C & \dots & F_v & \dots\end{bmatrix}$, $Z_F = \begin{bmatrix}Z_{F,C} & \dots & Z_{F,v} & \dots\end{bmatrix}$ as well as $Q = \begin{bmatrix}Q_C & \dots & Q_v & \dots\end{bmatrix}$. This allows the abbreviations in (21c) with $x_{\text{exo}}^{\top} = \begin{bmatrix}x_{\text{exo},C}^{\top} & \dots & x_{\text{exo},v}^{\top} & \dots\end{bmatrix}$:

$$\begin{aligned}K_{\text{exo},C} &= (-K\Pi_C + F_C + D^{\perp}Z_{F,C}),\\ K_{\text{exo},v} &= (-K\Pi_v + F_v + D^{\perp}Z_{F,v}).\end{aligned} \tag{25}$$

Lemma 2: If w given by exogenous system (4) with (5), (6) and u_{ss} as in (21c) as well as T as in (7) then

1) $\int_{t_0}^{t_0+T} w^{\top}w dt = T x_{\text{exo},0}^{\top}\widetilde{Q}^{\top}\widetilde{Q}x_{\text{exo},0}$

2) $\int_{t_0}^{t_0+T} u_{\text{ss}}^\top u_{\text{ss}} dt = T x_{\text{exo},0}^\top \widetilde{K}^\top \widetilde{K} x_{\text{exo},0}$
hold $\forall x_{\text{exo},0}$ with

$$\widetilde{Q} = \text{diag}\left(Q_C, \frac{1}{\sqrt{2}}\begin{bmatrix} Q_1 \\ Q_1 E_1 \end{bmatrix}, \dots, \frac{1}{\sqrt{2}}\begin{bmatrix} Q_{N_\Omega} \\ Q_{N_\Omega} E_{N_\Omega} \end{bmatrix}\right), \tag{26a}$$

$$\widetilde{K} = \text{diag}\left(K_{\text{exo},C}, \frac{1}{\sqrt{2}}\begin{bmatrix} K_{\text{exo},1} \\ K_{\text{exo},1} E_1 \end{bmatrix}, \dots\right) \text{ analogously and} \tag{26b}$$

$$E_v = \text{diag}\ \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots\right) \in \mathbb{R}^{2N_{\omega,v} \times 2N_{\omega,v}} \tag{26c}$$

incorporating (25).

Proof: Regarding (4) and $x_{\text{exo}}(\cdot) = e^{St} x_{\text{exo},0}$ gives

$$\int_{t_0}^{t_0+T} w^\top w dt = \int_{t_0}^{t_0+T} x_{\text{exo},0}^\top e^{S^\top t} Q^\top Q e^{St} x_{\text{exo},0} dt.$$

Considering (6) yields $e^{St} = \text{diag}(e^{S_C t}, \dots, e^{S_v t}, \dots)$ with

$$e^{S_C t} = I_{n_C}$$
$$e^{S_v t} = \text{ diag}\left(\begin{bmatrix} \cos(\omega_v t) & \sin(\omega_v t) \\ -\sin(\omega_v t) & \cos(\omega_v t) \end{bmatrix}, \dots\right)$$

and taking Lemma 1 1) and 2) into account, i.e., all off diagonal blocks - the interconnections of subsystems S_C, S_v $\forall v$ - vanish leading to

$$\int_{t_0}^{t_0+T} w^\top w dt = T x_{\text{exo},C,0}^\top Q_C^\top Q_C x_{\text{exo},0} + \sum_{v=1}^{N_\Omega} x_{\text{exo},v,0}^\top \int_{t_0}^{t_0+T} e^{S_v^\top t} Q_v^\top Q_v e^{S_v t} dt\ x_{\text{exo},v,0}. \tag{27}$$

Based on $e^{S_v t} = \cos(\omega_v t) I + \sin(\omega_v t) E_v$, it follows with Lemma 1 3), i.e., all cross terms $\cos(\omega_v t)\sin(\omega_v t)$ vanish after integration, and with Lemma 1 4) as well as (26c)

$$\begin{aligned} &\int_{t_0}^{t_0+T} e^{S_v^\top t} Q_v^\top Q_v e^{S_v t} dt \\ &= \int_{t_0}^{t_0+T} \cos^2(\omega_v t) Q_v^\top Q_v + \sin^2(\omega_v t) E_v^\top Q_v^\top Q_v E_v dt \\ &= \frac{T}{2}\left(Q_v^\top Q_v + E_v^\top Q_v^\top Q_v E_v\right). \end{aligned} \tag{28}$$

Substituting (28) in (27) and introducing (26a) finally gives 1). After including abbreviations (25), with

$$u_{\text{ss}}^\top u_{\text{ss}} = x_{\text{exo}}^\top \begin{bmatrix} K_{\text{exo},C} \\ K_{\text{exo},1} \\ \vdots \end{bmatrix} \begin{bmatrix} K_{\text{exo},C} & K_{\text{exo},1} & \dots \end{bmatrix} x_{\text{exo}}$$

2) can be easily derived in the same manner as in the previous case. ■

With the help of Lemma 2 it is possible to substitute (21b) and (4) in optimization problem (21) such that a linear SDP is accomplished. This is summarized in the following **main theorem**.

Theorem 2: Considering an exogenous system (4) with unknown, arbitrary initial value $x_{\text{exo},0}$ in block diagonal form (5) satisfying (6) and T given by (7) the following statement holds:
The optimization problem (21) is equivalent to the linear SDP

$$\min_{\gamma>0, Z_F, \Pi} \gamma \tag{29a}$$

subject to:

$$\begin{bmatrix} -\gamma \widetilde{Q}^\top \widetilde{Q} & \widetilde{K}^\top \\ \widetilde{K} & -\gamma I \end{bmatrix} \prec 0, \tag{29b}$$

$$\Pi S = (A - BK)\Pi + BF + BD^\perp Z_F \tag{29c}$$

with $\widetilde{Q}$ as in (26a) and $\widetilde{K}$ as in (26b) the latter depending affine linearly on optimization variables Π and Z_F through abbreviations (25).

Proof: It is feasible to apply Lemma 2. Therefore (21b) considering (21c) and (4) with initial value $x_{\text{exo},0}$ is equivalent to

$$T x_{\text{exo},0}^\top \widetilde{K}^\top \widetilde{K} x_{\text{exo},0} < \gamma^2 T x_{\text{exo},0}^\top \widetilde{Q}^\top \widetilde{Q} x_{\text{exo},0}.$$

As this inequality must hold $\forall x_{\text{exo},0}$ it equals $\widetilde{K}^\top \widetilde{K} \prec \gamma^2 \widetilde{Q}^\top \widetilde{Q}$. This can also be written as $-\gamma \widetilde{Q}^\top \widetilde{Q} - \widetilde{K}^\top \left(-\gamma I_{m(1+2N_\Omega)}\right)^{-1} \widetilde{K} \prec 0$. With $-\gamma I_{m(1+2N_\omega)} \prec 0$ the *Schur-Complement Lemma* is applicable, which finally yields (29b). Since the remaining constraints (21d) and objective function are independent of x_{exo} as well as $x_{\text{exo},0}$ and the exogenous dynamics (4) are independent of the optimization variables, (21e) can be neglected in the optimization. Therefore, (21) is equivalent to (29). ■

The solution of (29) gives an optimized prefilter $Z_{F,\text{opt}}$ which induces steady state evolution $\Pi_{\text{opt}} x_{\text{exo}}$ of the system resulting in minimal input-energy gain γ_{opt}. Thus, (29) accounts for controller synthesis regarding Section II-B problem III.

Since γ_{opt} is an upper bound of the energy $\int_{t_0}^{t_0+T} u_{\text{ss}}^\top u_{\text{ss}} dt$ by (21b), it may happen under certain circumstances, i.e., specific initial value $x_{\text{exo},0}$ that the consumed energy of optimized $u_{\text{ss,opt}}$ is higher than in case of $u_{\text{ss},+}$ disregarding the DoFs by choice $Z_{F,+} = 0_{m-p,n_{\text{exo}}}$. Therefore, an additional constraint can be incorporated into (29) to establish the energy of $u_{\text{ss},+}$ as an additional upper bound such that the energy of $u_{\text{ss,opt}}$ is always less or equal. This is a reasonable choice as indicated by the following discussion. It is well known, cf. [16], that $u_{\text{ss},+}$ is the solution of

$$\begin{aligned} &\min_{u(t)} u_{\text{ss},+}(t)^\top u_{\text{ss},+}(t) \\ &\text{subject to:} \\ &D u_{\text{ss},+}(t) = -M \Pi_+ x_{\text{exo}}(t) + N x_{\text{exo}}(t) \end{aligned} \tag{30}$$

with $x_{\text{ss},+} = \Pi_+ x_{\text{exo}}$ and $\Pi_+ S = (A - BD^+ M)\Pi_+ + BD^+ N$, thus minimizing the euclidean norm $\|u_{\text{ss},+}(t)\|$ regarding current point in time t. This is comparable to frequent results in control of *weakly redundant* systems, e.g., [4], [7].

Lemma 3: Under the same conditions as Lemma 2 and additionally $u_{\text{ss},+} = -K\Pi_+ x_{\text{exo}} + F x_{\text{exo}}$ requiring $\Pi_+ S = (A - BK)\Pi_+ + BF$ solvable, the following statements hold

1) $\int_{t_0}^{t_0+T} u_{\text{ss},+}^\top u_{\text{ss},+} dt = T x_{\text{exo},0}^\top \widetilde{K}_+^\top \widetilde{K}_+ x_{\text{exo},0}$ with $\widetilde{K}_+$ resulting from (26b) based on (25) with $Z_{F,C} = 0$, $Z_{F,v} = 0 \ \forall v$
2) With (26b) based on (25)

$$\int_{t_0}^{t_0+T} u_{\text{ss}}^\top u_{\text{ss}} dt \leq \int_{t_0}^{t_0+T} u_{\text{ss},+}^\top u_{\text{ss},+} dt \ \forall x_{\text{exo},0} \quad \text{(31a)}$$

$$\iff \begin{bmatrix} -\widetilde{K}_+^\top \widetilde{K}_+ & \widetilde{K}^\top \\ \widetilde{K} & -I \end{bmatrix} \preceq 0. \quad \text{(31b)}$$

Proof: Sketch: 1) can be derived in the same way as Lemma 2 1). Derivation of 2) can be adopted from derivation of (29b) in proof of Theorem 2 by applying *Schur-Complement Lemma* for nonstrict LMIs [14]. ■

Then including constraint (31b) into optimization problem (29) gives a linear SDP with a solution guaranteeing lower energy regarding $u_{\text{ss},+}$ and minimized input-energy gain γ.

Remark 3: In the previous discussion it was assumed that all inputs u_k with $k = 1, ..., m$ contribute equally to (21b). However, there might exist associated costs, e.g., from stricter input constraints or higher risk of actuator wear. Then, some actuators may be preferred to others. Hence, a weighting matrix $W \succ 0$ is introduced as follows

$$\int_{t_0}^{t_0+T} u_{\text{ss}}^\top W u_{\text{ss}} dt < \gamma^2 \int_{t_0}^{t_0+T} w^\top w dt.$$

Replacing (29b) in (29) by

$$\begin{bmatrix} -\gamma \widetilde{Q}^\top \widetilde{Q} & \widetilde{K}^\top \\ \widetilde{K} & -\gamma (I_{1+2N_\Omega} \otimes W)^{-1} \end{bmatrix} \prec 0 \quad \text{(32)}$$

gives the corresponding linear SDP. Same procedure can be applied for (31b) by including W into (30) yielding the weighted pseudoinverse based solution $u_{\text{ss},+,W}$ [16].

Remark 4: Optimization problem (29) is only solvable if a solution Π of (29c) exists. Since it is always possible to write $Z_F = \overline{Z}_K \Pi + \overline{Z}_F$, it follows the necessary and sufficient condition $\sigma(A - BK + BD^\perp \overline{Z}_K) \cap \sigma(S) \stackrel{!}{=} \emptyset$ [15]. With Ass. 2 it always exists a $\overline{Z}_K$ such that all eigenvalues in $\sigma(A - BK + BD^\perp \overline{Z}_K)$ lie in the open left half-plane; thus, (29c) is solvable. Furthermore, if (31b) is considered in (29) then solvability of $\Pi_+ S = (A - BK)\Pi_+ + BF$ is required; thus, $\sigma(A - BK) \cap \sigma(S) \stackrel{!}{=} \emptyset$ has to be satisfied.

V. SIMULATION RESULTS

To demonstrate the improvement in applications, the proposed approach is applied for the control of a F18 high alpha research vehicle. The system matrices are given by linearization in [2].

The system has $n = 4$ states $x = \begin{bmatrix} v & \alpha & \dot{\theta} & \theta \end{bmatrix}^\top$ and $p = 2$ outputs $y_1 = v$ and $y_2 = \theta$. These are longitudinal velocity v, angle of attack α and pitch angle θ. The input matrix satisfies rank $B = 3 > p$. With $m = 6 >$ rank B there are also *strongly redundant* inputs [11]. As can be easily seen by

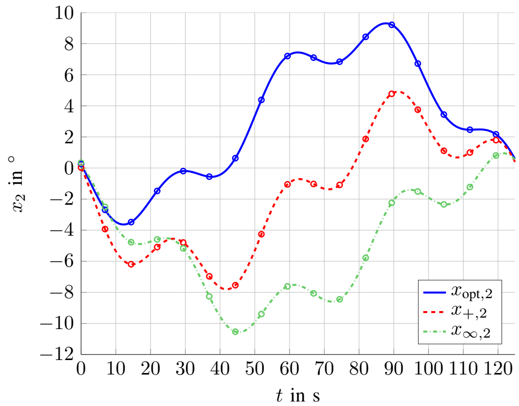

Fig. 1: Internal dynamics x_2 and related optimal trajectories (○ in matched color) parametrized by $e_2^\top \Pi x_{\text{exo}}$

the system structure it holds for $y_1 = x_1$ $\delta_1 = 1$, $y_2 = x_4$ and $\dot{y}_2 = x_3$ with $\delta_2 = 2$. Therefore a feasible representation of internal dynamics is $\eta = x_2$. It is desired to track the signals

$$w_1(t) = 20 \text{ km/h} + 20 \text{ km/h} \sin\left(\frac{0.05}{\text{s}} t + \frac{\pi}{4}\right)$$
$$w_2(t) = -17.5^\circ + 15^\circ \cos\left(\frac{0.05}{\text{s}} t\right) + 2.5^\circ \cos\left(\frac{0.2}{\text{s}} t\right) \quad \text{(33)}$$

with spectra $\Omega_1 = \{0.05\}$, $\Omega_2 = \{0.05, 0.2\}$ giving $\Omega = \{0.05, 0.2\}$ and $T = 125.7$ s. At stage of control design, the spectra are known but amplitudes and phases are not. A possible way to represent (33) for arbitrary amplitudes and phases is an exo-system (4) with $n_{\text{exo}} = 8$, $n_C = 2$ and subsystem matrices as in (6) with $\omega_1 = 0.05$, $\omega_2 = 0.2$.

With rank $D = p$ Ass. 1 holds. In addition, with $e_2^\top B D^\perp \neq 0$ the dynamics of x_2 are controllable, and thus Ass. 2 also holds. Consequently, application of Theorem 1 gives $u = -Kx + F x_{\text{exo}} + D^\perp (Z_K \tilde{x} + Z_F x_{\text{exo}})$ with K, F such that all eigenvalues of error dynamics in -2 and DoFs $Z_K \in \mathbb{R}^{4\times4}$, $Z_F \in \mathbb{R}^{4\times8}$. The DoFs Z_F are determined by solving (29) for $W = I$ and additional constraint (31b) giving the tracking control u_{opt}; the optimization was executed using [17] and [18]. Alternatively, the choice $Z_F = 0$ leads to $u_{\text{ss},+}$ given by (30).

Substituting $N x_{\text{exo}} = w^*$ in (13) and choosing $u_\infty = -Kx + D^+ w^* + D^\perp (Z_K^* x + Z_F^* w^*)$ allows to apply the proposed approach in [10], see Remark 2, which cannot incorporate any information about the desired values. Hence, complete substitution by w^* is required. Then by formulating the transfer function $G_{u_\infty, w^*}(s)$, the optimization in [10] gives the DoFs Z_K^* and Z_F^* as arguments of the minimum of $\|G_{u_\infty, w^*}(j\omega)\|_\infty$.

On the next page, Tab. I summarizes the results for the different control laws. It might be stated that minimization of $\|G_{u_\infty, x_{\text{exo}}}(j\omega)\|_\infty$ yields the same results for the DoFs in u_∞ which give the minimum $\|G_{u_\infty, w^*}(j\omega)\|_\infty = 2.5951$. In case of u_+ and u_∞, (29) was solved as a feasibility problem to obtain γ - find a feasible γ for given $Z_{F,+} = 0$ or $Z_{F,\infty} = Z_K^* \Pi_\infty + Z_F^*$, respectively.

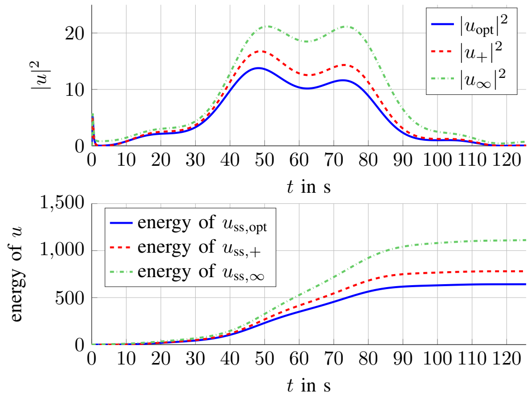

Fig. 2: Squared euclidean norm and input energy for one period T

Clearly, u_∞ guarantees the smallest H_∞-norm but does not improve the performance goal γ. On the other hand, u_{opt} given by optimization lowers γ^2 by 10%. Nonetheless, γ gives only information about the highest singular value, and thus simulation results will also depend on chosen initial values of the exo-system. In case of u_{opt} and u_+, the remaining Z_K is calculated by implementing a linear quadratic regulator for $\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$ with $\tilde{u} = \left(-K + D^\perp Z_K\right)\tilde{x}$ based on (20) and yielding the eigenvalues of the closed loop $\sigma\left(A - BK + BD^\perp Z_K\right) = \{-2, -2, -2, -1.9157\}$. In case of u_∞, it comes to $\sigma\left(A - BK + BD^\perp Z_K^*\right) = \{-2, -2, -2, -0.2814\}$ with Z_K^* already given by optimization and internal dynamics significantly slower.

TABLE I: Results for different control laws

	u_{opt}	u_+	u_∞
γ	8.9927	9.4715	9.4592
$\Vert G_{\cdot,x_{\text{exo}}}(j\omega)\Vert_\infty$	81.4312	66.6909	18.0379
ω_∞	1.9157	2	0.5551

The results for tracking (33) for initial values $x_0 = 0$ are now considered. Regarding output dynamics, all control laws yield the same tracking behavior. In contrast, Fig. 1 shows completely different optimal trajectories which emphasizes the influence of the internal dynamics on energy of control inputs and its importance for optimization in control of *over-actuated* systems. Finally, Fig. 2 proofs the effectiveness of the new approach. The norm of the input u_{opt} is remarkably lowered and the energy is reduced by 17.71% regarding $u_{\text{ss},+}$ and by even 42.22% regarding the H_∞-norm based optimization. Furthermore, energy of u_+, which is an upper bound of related energy of u_{opt} through (31b), is 29.79% smaller than of u_∞; hence, the method by [10] leads to a relevant performance decrease. This indicates that taking all provided information on reference signals into account is necessary to guarantee a performance increase.

VI. Conclusion

In this contribution an optimization procedure for decoupling of *over-actuated* linear systems and sinusoidal references with frequencies exclusively known was introduced. It was shown that additional DoFs arise and a suitable parametrization based on a state transformation was found. In this way, the tasks of stabilizing internal dynamics, control of the transient behavior respectively, and optimizing input-energy gain for steady state evolution can be separated. Accounting for optimization, a linear SDP was derived. An additional constraint assures less consumed energy of control inputs than in the non-optimized case independent of exogenous initial values, and thus overcomes shortcomings of other approaches. Finally, an example of an *over-actuated* aircraft proved the effectiveness of the proposed approach compared to previous results.

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