Static Optimal Decoupling Control for Linear Over-Actuated Systems Regarding Time-Varying References

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***Abstract*— We address static decoupling control for linear *over-actuated* systems and time-varying references given by exogenous systems with arbitrary eigenvalues. Based on mild assumptions, additional degrees of freedom in form of an input are provided. Then an optimal tracking problem for quadratic integral cost is formulated. Despite the time dependency of the cost and dynamics, we derive a static feedback and pre-filter satisfying necessary optimality conditions for infinite final time. These can be calculated by the solution of an algebraic Riccati equation and a Sylvester equation, respectively. In spite of its simplicity in derivation as well as implementation - offering great convenience for practical use - we prove optimal transient behavior to a unique optimal stationary trajectory of the system states. Or, more precisely, of the internal dynamics which are proven to exist. Moreover, the static control law is verified to be a close approximation of the computationally expensive finite time optimal solution if simple qualitative criteria are met. An application to a helicopter model reveals the high efficiency of our approach compared to others.**

I. Introduction

The majority of systems in safety-critical processes are *over-actuated*, mainly, to overcome actuator faults or constraints. These include aircraft [18], automobiles [6], chemical processes [14] and many more. The question arises: how can the additional actuators be beneficially exploited during proper operation?

In this regard, we consider LTI-systems in the form of

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{1}$$

with states $x \in \mathbb{R}^n$, inputs $u \in \mathbb{R}^m$, outputs $y \in \mathbb{R}^p$ which are *over-actuated* by $m > p$. Most significantly, it holds

$$p < \text{rank}(B) \leq m. \tag{2}$$

Since virtual control inputs leading to a square system cannot be introduced, the fundamental assumption of *control allocation* (CA), see [12], [13], is violated. E.g., non-negligible actuator dynamics lead to (2) for $m > p$. On the contrary, (2) can also be induced on purpose by disregarding outputs of minor importance. This could be valuable if meaningful trajectories are unknown or hard to compute, if the exo-system order should be reduced or if the closed loop is desired to perform at minimized costs. Despite the motivation this procedure is not popular which might be caused by the lack of suitable methods.

In contrast to the case $\text{rank}(B) = p$ of CA, the definition of additional degrees of freedom (DOF) for optimization is not unique here. Hence, the DOF are usually determined with respect to a certain control method. So far, this led to conditions for a cost decrease in standard linear quadratic regulator (LQR) problems (no reference signals) [9], an optimized choice of a stationary solution of the *Francis' equations* [16] or the design of an "inverse model allocation" in output regulation [20], etc. A control-method-independent definition of DOF by a dynamic allocator is given in [7] but optimization-wise only constant references are considered.

Our focus is on static control methods that guarantee row-by-row decoupling of the path of exogenous references w to system outputs y which is often desired in practical applications like [14], [18]. Then, the change of reference of one output does not affect the others at all, in particular during transition. The same holds at initialization for any initial values of the plant. As a result, the behavior of the closed loop is easy to predict and the static control can be implemented at minimal computational cost.

Regarding *over-actuated* systems, most contributions are concerned with sufficient and necessary conditions for static decoupling, e.g. [5]. They typically do not consider how additional DOF, most likely existing, could be used once the system has been decoupled. This is crucial since an *over-actuated* system should operate efficiently. In this context, many simply apply CA by minimizing $\|u\|_2$ and neglecting the influence on internal dynamics, e.g. [14], [18]. Or they minimize upper bounds like the input energy gain γ with $\int_{t_0}^{\infty} u^\top R u \; \mathrm{d}t < \gamma^2 \int_{t_0}^{\infty} w^\top w \; \mathrm{d}t$ [22] unable to include any information on the references w (even $w = const.$). The latter was significantly improved for sinusoidal references in [3]. However, this is still a special case.

In view of the decoupling constraint leading to $u = g(x, z, t)$, we consequently consider the minimization of general quadratic performance indexes

$$J = \frac{1}{2} \int_{t_0}^{t_1} x^\top Q x + u^\top R u \; \mathrm{d}t \tag{3}$$

with $Q \succeq 0$, $R \succ 0$ by means of the "new" control input z, i.e. the DOF induced by (2). Hence, we intend to minimize the weighted energy of the states of the internal dynamics and the weighted input energy. Notice that this does not lead to a standard LQR problem since $u = g(x, z, t)$ is required to guarantee the decoupling of output dynamics.

Based on mild assumptions in Section III, we will see that a static row-by-row decoupling control exists and DOF $z \in \mathbb{R}^{m-p}$ can be introduced without requiring additional dynamics in the control loop in contrast to [7], [20]. For the first time, as far as we know, we address our specific goal by formulating an optimal control problem based on (3) in

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Section IV. It is conveniently formulated in the LQ tracking framework though not completely standard [1], [17].

Despite the time-varying references and cost (3), we will be able to deduce a **static optimal control** for z in case $t_1 \to \infty$ and any initial conditions in Section IV-B. A similar insight was stated in [15] but did not draw much attention which may be reasoned by the following. In contrast to [15], our results will satisfy the necessary conditions [11] and will allow a discussion on optimality in Section IV-B. Furthermore, we give qualitative criteria for the quality of approximation of the computationally expensive optimal solution for finite t_1.

Finally, we investigate the applicability and performance in simulations of a helicopter model in Section V. Based on a thoroughly discussion, we compare our results with the methods of [14], [18], [22] in Example 1 and to the proposed optimization of [16], [20, Sec. V. B] in Example 2.

Mathematical notations: The zero matrix 0 and identity matrix I have appropriate dimensions if not denoted explicitly, e.g. $0_{a,b} \in \mathbb{R}^{a\times b}$. We write $M^{\perp}$ for the basis of the null space of a matrix, M^{+} for its *Moore-Penrose* pseudoinverse and use $\mathrm{He}(M) = M + M^{\top}$ with M real as well as $M = \mathrm{diag}(\dots)$ for the definition of a diagonal matrix. A symmetric matrix is positive (positive semi-) definite if $M \succ (\succeq)\ 0$. The spectrum is denoted by $\sigma(M)$. The vector e_i of appropriate dimensions has a one in the i-th row, zeros else.

II. PRELIMINARIES

A. System setup and exogenous inputs

Regarding (1), pair (A, B) is controllable and $\mathrm{rank}(C) = p$ holds. The reference inputs w are given by an exo-system

$$\begin{aligned} \dot{x}_{\text{exo}} &= S x_{\text{exo}}, \\ w &= O x_{\text{exo}} \end{aligned} \tag{4}$$

with $x_{\text{exo}} \in \mathbb{R}^{n_{\text{exo}}}$, $w \in \mathbb{R}^p$. Wlog, we assume $\mathrm{Re}(\lambda_{\text{exo},i}) \geq 0\ \forall \lambda_{\text{exo},i} \in \sigma(S)$ since asymptotically stable eigenvalues do not contribute to the stationary behavior and can be disregarded [21].

B. Problem statement

Our goal is to find a control law for system (1) that

I) decouples (1) by diagonalization of $G_{w\to y}(s)$,

II) leads to asymptotic tracking of $w(\cdot)$, if desired,

III) solves an optimal control problem based on quadratic performance indexes such as (3)

IV) is static, satisfies necessary conditions for $t_1 \to \infty$ [11] and approximates the finite time solution of III)

Problem I), II) are approached in the next section followed by III), IV) in Section IV.

III. STATIC DECOUPLING FOR LINEAR SYSTEMS

Here, we formulate a row-by-row decoupling control law introducing additional degrees of freedom (DOF). This will be the basis of the following sections. The derivation is mainly based on [10] having similarities to [3], [22].

With (A, B) controllable and $\mathrm{rank}(C) = p$ a nonsingular transformation matrix $T_D \in \mathbb{R}^{n\times n}$ exists such that

$$q = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ \vdots \\ y_1^{(\delta_1-1)} \\ y_2 \\ \vdots \\ \eta \end{bmatrix} = \begin{bmatrix} c_1^\top \\ c_1^\top A \\ \vdots \\ c_1^\top A^{\delta_1-1} \\ c_2^\top \\ \vdots \\ T_\eta \end{bmatrix} x = T_D x \tag{5}$$

with the relative degree $\delta_i = \min_j\{j \in \mathbb{N}^+ : c_i^\top A^{j-1} B \neq 0\}$ of output y_i defined as for square systems. The relative degree of system (1) is $\delta = \sum_{i=1}^{p} \delta_i \leq n$, see [10]. The state vector of the internal dynamics is $\eta \in \mathbb{R}^{n-\delta}$. Now, it is feasible to accomplish

Lemma 1: For system (1) with condition (2) being satisfied it holds $\delta \leq n - (\mathrm{rank}(B) - p)$.

Proof: Since $\delta \leq n$, let us assume $n - (\mathrm{rank}(B) - p) < \delta$. It follows for the dimension of the internal dynamics: $n_\eta = n - \delta < \mathrm{rank}(B) - p$. Thus, the application of the transformation (5) to the input matrix of (1) gives

$$\begin{aligned} T_D B = \Big[0_{\delta_1-1,m}^\top \quad & \left(c_1^\top A^{\delta_1-1} B\right)^\top \quad \dots \\ & 0_{\delta_p-1,m}^\top \quad \left(c_p^\top A^{\delta_p-1} B\right)^\top \quad B_\eta^\top \Big]^\top \end{aligned}$$

with $B_\eta \in \mathbb{R}^{n_\eta \times m}$. Apparently, it holds $\mathrm{rank}(T_D B) \leq p + n_\eta < \mathrm{rank}(B)$. This is a contradiction since T_D nonsingular implies $\mathrm{rank}(T_D B) = \mathrm{rank}(B)$. ■

Based on Lemma 1, we can conclude that the internal dynamics $\eta \in \mathbb{R}^{n-\delta}$ always exist, i.e. $n - \delta \neq 0$, and are of order $n_\eta \geq \mathrm{rank}(B) - p$. This is important since these are affected by the DOF z as we will see.

In order to find a decoupling control, we merge $y_i^{(\delta_i)} = c_i^\top A^{\delta_i} x + c_i^\top A^{\delta_i - 1} B u$ for all i. Then, if u satisfies

$$\underbrace{\begin{bmatrix} c_1^\top A^{\delta_1-1} B \\ \vdots \\ c_p^\top A^{\delta_p-1} B \end{bmatrix}}_{D} u = \begin{bmatrix} -c_1^\top A^{\delta_1} \\ \vdots \\ -c_p^\top A^{\delta_p} \end{bmatrix} x + \nu \tag{6}$$

with decoupling matrix $D \in \mathbb{R}^{p\times m}$ and the virtual input $\nu \in \mathbb{R}^p$ we have row-by-row decoupled dynamics $y^{(\delta_i)} = \nu_i$ for each output. By introducing

$$\nu_i = \alpha_{i,\delta_i} w_i^{(\delta_i)} - \sum_{j=0}^{\delta_i-1} a_{i,j} (y_i^{(j)} - \alpha_{i,j} w_i^{(j)}) \tag{7}$$

with $\alpha_{i,j} \in \{0, 1\}$, we achieve the dynamics

$$\tilde{y}_i^{(\delta_i)} + \sum_{j=0}^{\delta_i-1} a_{i,j} \tilde{y}_i^{(j)} = 0 \tag{8}$$

with $\tilde{y}_i^{(j)} = y_i^{(j)} - \alpha_{i,j} w_i^{(j)}$. For $\alpha_{i,j} = 1\ \forall j$ this leads to asymptotically stable error dynamics $\tilde{y}_i = y_i - w_i$ of the i-th output which needs $w_i(\cdot)$ to be δ_i-times differentiable (satisfied by (4)). On the contrary, when asymptotic tracking is not required, we simply choose $\alpha_{i,0} \neq 0$, $\alpha_{i,j} = 0$ else.

After substitution of $y_i^{(j)}$ by transformation (5) and of $w^{(i)}$ by the help of (4) with $w_i = o_i^\top x_{\text{exo}}$ in (7), the elements ν_i of the virtual input ν in (6) can be given by

$$\nu_i = \sum_{j=0}^{\delta_i - 1} -a_{i,j} c_i^\top A^j x + o_i^\top \left(\alpha_{i,\delta_i} S^{\delta_i} + \sum_{j=0}^{\delta_i - 1} \alpha_{i,j} a_{i,j} S^j \right) x_{\text{exo}}.$$

Substituting ν in (6), assembling all terms related to x as well as x_{exo} and introducing suitable matrices $M \in \mathbb{R}^{p \times n}$ and $N \in \mathbb{R}^{p \times n_{\text{exo}}}$, respectively, finally leads to

$$Du = -Mx + Nx_{\text{exo}}. \tag{9}$$

At this point, it is important to notice that explicit necessary and sufficient conditions for the row-by-row decoupling by static feedback of system (1) in case of (2) are not known yet, c.f. [5]. Typically, sufficient conditions are applied as in [22], [3]. This leads to the following mild assumptions motivated by the decoupling of square systems [10].

Assumption 1: It holds $\text{rank}(D) = p$.

Then we can give the right inverse $D^+ = D^\top \left(DD^\top\right)^{-1}$ with $DD^+ = I_{p,p}$; thus, there exists a u satisfying (9). Based on the *Rank-Nullity-Theorem*, we denote the $m - p$ dimensional null space of D by $\text{null}(D) = \text{span}\left(D^\perp\right)$ with $D^\perp \in \mathbb{R}^{m \times (m-p)}$. It holds $\text{rank}(D^\perp) = m - p$ and $DD^\perp = 0_{p,m-p}$.

With the following assumption we ensure that any unstable output-decoupling zero is not invariant. Hence, all invariant zeros of system (1) lie in the open left half-plane, cf. [22].

Assumption 2: The pair $\left(A - BK, BD^\perp\right)$ with K satisfying $DK = M$ is stabilizable.

Remark 1: Since $m > p$, the assumptions are clearly less restrictive than in the square case $m = p$, cf. [10]. In particular, Asmp. 2 most likely holds because system (1) with condition (2) "almost always" has no invariant zeros [8].

Finally, we are ready to give a summarizing lemma

Lemma 2 (cf. [3]): Let system (1) be *over-actuated*, i.e. (2) holds, and Asmp. 1, 2 are satisfied. With the decoupling matrix D, matrices M and N as in (9) and a suitable choice of $a_{i,j}$ with $i = 1, \ldots, p$ and $j = 0, \ldots, \delta_i - 1$ such that (8) is asymptotically stable $\forall i$, the control law

$$u = -Kx + Fx_{\text{exo}} + D^\perp z \tag{10}$$

with

$$\begin{aligned} K &= D^+ M, \\ F &= D^+ N \end{aligned} \tag{11}$$

diagonalizes $G_{w \to y}(s)$; hence, it ensures row-by-row decoupling. Furthermore, the input $z \in \mathbb{R}^{m-p}$ allows stabilizing the pair $\left(A - BK, BD^\perp\right)$, in particular the internal dynamics which are invisible at the outputs. ■

As we can see Lemma 2 gives a solution of Problems I), II) and, moreover, provides explicitly a "new" input z which represents the DOF induced by (2). Since $DD^\perp = 0$, it does not affect the output dynamics similar to [7], [20] but in contrast additional controller dynamics are not necessary. In the next section we will see how optimal control theory can be used to account for III) and IV).

IV. MINIMIZATION OF QUADRATIC PERFORMANCE INDEXES

In order to use the DOF z for minimizing the input energy and/or for keeping the states of internal dynamics small, we solve an optimal control problem regarding III) in Section IV-A. The influence of z on the internal dynamics is treated implicitly in order to support the modularity of our approach. In spite of the time-varying references, we show in part B for the first time that a static control law arises for infinite final time $t_1 \to \infty$ which satisfies necessary optimality conditions. This allows us to analyze optimality of the static control law after separating transient and stationary behavior. Besides, approximative properties in view of the finite time optimal solution are examined. Eventually, we state our main result.

A. Formulation and solution of an optimal control problem

Applying control law (10) to system (1) and cost (3), which allows us to penalize the energy of the states of the internal dynamics and of the inputs, we formulate

Optimal Control Problem A: Find an optimal control $z^*(t)$ on $t \in [t_0, t_1]$ which minimizes the cost functional

$$J(z(\cdot)) = \frac{1}{2} \int_{t_0}^{t_1} \begin{bmatrix} x \\ z \\ x_{\text{exo}} \end{bmatrix}^\top \begin{bmatrix} \widetilde{Q} & \widetilde{S} & S_w \\ \widetilde{S}^\top & \widetilde{R} & R_w \\ S_w^\top & R_w^\top & Q_w \end{bmatrix} \begin{bmatrix} x \\ z \\ x_{\text{exo}} \end{bmatrix} \mathrm{d}t$$

with $\widetilde{Q} = Q + K^\top RK$, $\widetilde{S} = -K^\top RD^\perp$, $\widetilde{R} = D^{\perp\top} RD^\perp$, $S_w = -K^\top RF$, $R_w = D^{\perp\top} RF$ and $Q_w = F^\top RF$ for $Q \succeq 0$, $R \succ 0$ regarding the decoupled system dynamics

$$\dot{x} = f(x, z, t) = \widetilde{A}x + \widetilde{B}z + B_w x_{\text{exo}} \tag{12}$$

with $\widetilde{A} = A - BK$, $\widetilde{B} = BD^\perp$, $B_w = BF$ with initial value $x(t_0) = x_0$ and exo-dynamics (4) with $x_{\text{exo}}(t_0) = x_{\text{exo},0}$.

This is a fixed final time, free end-point problem without terminal-costs, cf. [2]. Furthermore, the integrand of $J(z(\cdot)) = \int_{t_0}^{t_1} L(x, z, t)\mathrm{d}t$ and the dynamics (12) are explicitly time-dependent with regard to $x_{\text{exo}}(t)$. Due to the occurring cross-terms $x^\top \widetilde{S} z$, $x_{\text{exo}}^\top R_w^\top z$ and the influence of x_{exo} via B_w on the dynamics, *Opt. Prob. A* differs from the linear quadratic tracking problems considered in standard literature, e.g. [1], [17]. Its solution is given by

Lemma 3: The optimal control of *Opt. Prob. A* is

$$z^* = -\widetilde{K}(t)x - \widetilde{R}^{-1} \left(\widetilde{B}^\top v(t) + R_w x_{\text{exo}} \right) \tag{13}$$

with $\widetilde{K}(t) = \widetilde{R}^{-1}(\widetilde{S}^\top + \widetilde{B}^\top P(t))$, $0 \preceq P(t) : \mathbb{R} \to \mathbb{R}^{n \times n}$ and $v(t) : \mathbb{R} \to \mathbb{R}^n$ being obtained from

$$-\dot{P} = \text{He}\left(P\widetilde{A}\right) - \left(P\widetilde{B} + \widetilde{S}\right) \widetilde{R}^{-1} \left(P\widetilde{B} + \widetilde{S}\right)^\top + \widetilde{Q} \tag{14}$$

for final value $P(t_1) = 0$ as well as from

$$-\dot{v} = \left(\widetilde{A} - \widetilde{B}\widetilde{K}\right)^\top v + \left(P\widetilde{B}_w + S_w - \widetilde{S}\widetilde{R}^{-1} R_w\right) x_{\text{exo}} \tag{15}$$

with $\widetilde{B}_w = -\widetilde{B}\widetilde{R}^{-1} R_w + B_w$ and for final value $v(t_1) = 0$.

Proof: We examine the necessary conditions by standard means. Regarding sufficient conditions, we have to treat our problem individually.

Using the sweep method $\lambda = Px+v$ as in [17], it is easy to check that $-\dot{\lambda} = \frac{\partial H}{\partial x}$ with transversality condition $\lambda(t_1) = 0$ is satisfied and z^* is a unique global minimum of the Hamiltonian $H(x^*, z^*, t) = L(x, z, t) + \lambda^\top f(x, z, t)$ since $\partial H/\partial z = 0$ and $R \succ 0$. This requires that $P(t)$, $v(t)$ exist for all $t \in [t_0, t_1]$. Since in our case $J(z(\cdot)) \geq 0 \; \forall x_{\text{exo}} \in \mathbb{R}^{n_{\text{exo}}}$, we can prove $\widetilde{Q} - \widetilde{S}\widetilde{R}^{-1}\widetilde{S}^\top \succeq 0$ by *Schur-Complement-Lemma* (choose $x_{\text{exo}}(t) = 0$ wlog). Hence, $P(t) \succeq 0$ always exists, follow [1, p. 23 with adaption for cross-terms p. 57]. The existence of $v(t)$ follows immediately. Consequently, the necessary conditions given by *Pontryagin's Minimum Principle*, e.g. [2, Thm. 5-10], are satisfied.

In contrast to the standard linear quadratic tracking problem, we cannot simply deduce optimality by $Q \succeq 0$, $R \succ 0$. Nonetheless, by introducing variations as in [2], e.g. $x(t) = x^*(t) + \epsilon\delta x(t)$ with $\epsilon \in \mathbb{R}$, we see that the second variation of J, i.e. $\delta^2 J = \frac{d^2 J(z^* + \epsilon\delta z)}{d\epsilon^2}\big|_{\epsilon=0}$, is bounded below:

$$\delta^2 J \geq \int_{t_0}^{t_1} \left(-K\delta x + D^\perp \delta z\right)^\top R \left(-K\delta x + D^\perp \delta z\right) \mathrm{d}t.$$

Taking (11) into account, $-K\delta x + D^\perp \delta z = 0$ always implies $\delta z = 0$. Consequently, $\delta^2 J > 0 \; \forall \delta z(\cdot) \not\equiv 0$ because $R \succ 0$ and optimality of z^* follows by sufficiency [2]. ■

Regarding (13), we observe that the optimal control is always time-varying even for $x_{\text{exo}} \equiv 0$, see [17]. In addition, it can only be applied to a specific $x_{\text{exo},0}$ and exactly known t_1. For practical implementation, it has to be calculated in advance, e.g. by backward integration of (14), (15) as suggested by [17]. Then, time series $P(t)$, $v(t)$ have to be stored and interpolated on-line which is computationally expensive.

B. The infinite time case: a static control law

The disadvantages of the optimal control (13) motivates us to analyze the infinite time case, $t_1 \to \infty$. For simplicity, we firstly disregard unstable exo-systems but give the needed extensions in a remark later on. Now, let us define the solutions $P(t, t_1)$ of (14) and $v(t, t_1)$ of (15) for a given final value at time t_1, i.e. $P(t_1) = 0$ and $v(t_1) = 0$. Then:

Theorem 1: If Asmp. 2 and $\mathrm{Re}(\lambda_{\text{exo},i}) = 0 \; \forall \lambda_{\text{exo},i} \in \sigma(S)$ are satisfied, it holds $\lim_{t_1 \to \infty} P(t, t_1) = P_s \succeq 0$ with algebraic Riccati equation (ARE)

$$\mathrm{He}\left(P_s\widetilde{A}\right) - \left(P_s\widetilde{B} + \widetilde{S}\right)\widetilde{R}^{-1}\left(P_s\widetilde{B} + \widetilde{S}\right)^\top + \widetilde{Q} = 0 \quad (16)$$

and $\lim_{t_1 \to \infty} v(t, t_1) = \Pi_v x_{\text{exo}}(t)$ with Sylvester equation

$$\Pi_v S = -\left(\widetilde{A} - \widetilde{B}\widetilde{K}_s\right)^\top \Pi_v - \left(P_s\widetilde{B}_w + S_w - \widetilde{S}\widetilde{R}^{-1}R_w\right) \quad (17)$$

with $\widetilde{K}_s = \widetilde{R}^{-1}(\widetilde{S}^\top + \widetilde{B}^\top P_s)$, $\widetilde{B}_w$ as in Lemma 3. Furthermore, the control law given by (13) becomes time-invariant:

$$z_s^* = -\widetilde{K}_s x - \widetilde{R}^{-1}\left(\widetilde{B}^\top \Pi_v + R_w\right) x_{\text{exo}} \quad (18)$$

and if the pair $(\widetilde{A} - \widetilde{B}\widetilde{R}^{-1}\widetilde{S}^\top, \widetilde{W})$ with any $\widetilde{W}^\top\widetilde{W} = \widetilde{Q} - \widetilde{S}\widetilde{R}^{-1}\widetilde{S}^\top$ is detectable, the closed loop with system matrix $(\widetilde{A} - \widetilde{B}\widetilde{K}_s)$ is asymptotically stable. In addition, necessary optimality conditions for $t_1 \to \infty$ [11] are satisfied.

Proof: The first part is standard. With $\widetilde{Q} - \widetilde{S}\widetilde{R}^{-1}\widetilde{S}^\top \succeq 0$ and Asmp. 2, it follows by [1, p. 37-40 with adaption for cross-terms p. 57] that $\lim_{t_1 \to \infty} P(t, t_1) = P_s \succeq 0$ exists and is uniquely given by (16). Based on the detectability, asymptotic stability of the closed loop follows by [1, p. 58].

According to [1, p. 85], the solution of (15) follows from

$$\lim_{t_1 \to \infty} v(t, t_1) = \lim_{t_1 \to \infty} \int_t^{t_1} \mathrm{e}^{(\widetilde{A} - \widetilde{B}\widetilde{K}_s)^\top(\tau - t)} \cdot \left(P_s\widetilde{B}_w + S_w - \widetilde{S}\widetilde{R}^{-1}R_w\right) x_{\text{exo}} \, \mathrm{d}\tau. \quad (19)$$

Based on geometric methods, e.g. as in output regulation theory [21], we replace the term in brackets with the help of (17). Then the integration yields $v(t, t_1) = -e^{\left(\widetilde{A} - \widetilde{B}\widetilde{K}_s\right)^\top (t_1 - t)} \Pi_v x_{\text{exo}}(t_1) + \Pi_v x_{\text{exo}}(t)$. Observe that Π_v always exists because in our case $\sigma(\widetilde{A} - \widetilde{B}\widetilde{K}_s) \cap \sigma(S) = \emptyset$, cf. [21]. With $(\widetilde{A} - \widetilde{B}\widetilde{K}_s)$ asymptotically stable and $\mathrm{Re}(\lambda_{\text{exo},i}) = 0 \; \forall \lambda_{\text{exo},i} \in \sigma(S)$, the first term converges to zero for $t_1 \to \infty$ and any given t. Finally, we have $\lim_{t_1 \to \infty} z^* = z_s^*$.

Based on the convergence, z_s^* satisfies necessary conditions in [11] which follow from the first part of the proof of Lemma 3 by discarding the transversality condition. ■

Compared to [15], this new result is important since necessary conditions are met; hence, the aspects of optimality of z_s^* can be deduced as done in Section IV-C.

As can be seen the control in the infinite time case overcomes the disadvantages mentioned in the last section. However, we cannot deduce optimality for (18) directly because in general J will be unbounded for any z in case of $t_1 \to \infty$. Nonetheless, let us firstly adapt the argumentation [2, for constant references in Sec. 9-10] to our case. Assuming t_1 be "exceedingly" large we give

$$t_0 \ll t_v \ll t_P \ll t_1.$$

Following Theorem 1, $P(t)$ converges to P_s for t_1 increasing, see also discussion [2, p. 772]. Since $t_P \ll t_1$ we can approximate $P(t) \approx P_s$ for $t \in [t_0, t_P]$. Regarding (15) by utilizing the approximation we calculate

$$v(t) \approx \mathrm{e}^{\left(\widetilde{A} - \widetilde{B}\widetilde{K}_s\right)^\top (t_v - t)} (v(t_v, t_P) - \Pi_v x_{\text{exo}}(t_v)) + \Pi_v x_{\text{exo}}$$

for $t \leq t_v$. Based on the proof above it approximately holds $v(t_v, t_P) \approx \Pi_v x_{\text{exo}}(t_v)$ for $t_v \ll t_P$ and we can neglect the first term for $t \leq t_v$ yielding $v(t) \approx \Pi_v x_{\text{exo}}$ for $t \in [t_0, t_v]$. Concluding $P(t) \approx P_s$ and $v(t) \approx \Pi_v x_{\text{exo}}(t)$ gives (18) as an approximation of (13) for $t \in [t_0, t_v]$. This is basically similar to the result in [2, p. 803] but with the essential difference that we handled time-varying references here.

Contrary to [2], we also want to apply (13) on the complete interval $[t_0, t_1]$. This will lead to a performance decrease on $[t_v, t_1]$. But the length of $[t_P, t_1]$ depends inversely proportional on the rate of convergence 2α associated with $P(t) \to P_s$, see [4], and similarly depends $t_P - t_v$ on $\alpha - \max_i(\mathrm{Re}(\lambda_{\text{exo},i})) > 0$.

As a consequence, we are able to establish the qualitative criteria: the time-invariant control (18) will approximate the optimal solution (13) on $\left[t_0, t_1\right]$ increasingly closer the larger the final time t_1 and the higher the rate of convergence α are. This knowledge is very valuable since α equals the degree of stability of the closed loop which can be easily increased as explained in the remarks below. In addition, we consider a tracking problem rather than a transient problem. Thus, t_1 can be assumed large compared to the dominant time-constant of (4). Hence, we can assume that (18) approximates (13) sufficiently well and leads to an adequate performance in many cases. In Section V, we will see that this can be true even for moderately large t_1 and α.

Remark 2: A certain degree of stability, $\mathrm{Re}(\lambda_{\eta,i}) \leq -\alpha$ $\forall i = 1, \ldots, n_\eta$, of internal dynamics might be desired. We apply the proposed method in [1, Sec. 3.5]. With $\widetilde{I} = T_D^{-1}\mathrm{diag}(0_{\delta,\delta}, I_{n_\eta,n_\eta})T_D$ assume $(\widetilde{A} + \alpha\widetilde{I}, \widetilde{B})$ stabilizable and $(\widetilde{A} + \alpha\widetilde{I} - \widetilde{B}\widetilde{R}^{-1}\widetilde{S}^\top, \widetilde{W})$ with any $\widetilde{W}^\top\widetilde{W} = \widetilde{Q} - \widetilde{S}\widetilde{R}^{-1}\widetilde{S}^\top$ detectable. Then replacing $\widetilde{A}$ by $\widetilde{A} + \alpha\widetilde{I}$ in (16) gives the solution P_s^* with $\widetilde{K}_s^*$ guaranteeing asymptotic stability of $\widetilde{A} + \alpha\widetilde{I} - \widetilde{B}\widetilde{K}_s^*$. Now modifying *Opt. Prob. A* by $\widetilde{Q} = Q + 2\alpha P_s^* + K^\top R K$, the application of Theorem 1 gives the solution $P_s = P_s^*$ of (16) and it results $K_s = K_s^*$. Hence, application of Theorem 1 under these modifications of *Opt. Prob. A* guarantees $\mathrm{Re}(\lambda_{\eta,i}) \leq -\alpha$. Then, by proper choice of $a_{i,j}$ in Lemma 2 it holds $\mathrm{Re}(\lambda_i) \leq -\alpha\ \forall i$.

Remark 3: Exploiting Remark 2, we can apply Theorem 1 even when $\lambda_{\mathrm{exo},i} \in \sigma(S)$ with $\mathrm{Re}(\lambda_{\mathrm{exo},i}) > 0$ exist. By choosing $\alpha > \max_j \mathrm{Re}(\lambda_{\mathrm{exo},j})$, $\mathrm{Re}(\lambda_i) < -\max_j \mathrm{Re}(\lambda_{\mathrm{exo},j})$ is guaranteed and $\lim_{t_1\to\infty} v(t, t_1) = \Pi_v x_{\mathrm{exo}}$ can be proven with Π_v existing since $\sigma(-(\widetilde{A} - \widetilde{B}\widetilde{K}_s)) \cap \sigma(S) = \emptyset$.

C. Optimal transients and stationary trajectories

In contrast to [15], our results in Theorem 1 satisfy the necessary optimality conditions for infinite final time [11]. Hence, we are able to show that these are optimal in specific regards. For the analysis it is beneficial to separate transient and stationary behavior, e.g. as can be seen in [23] in the simpler case of constant references.

We apply the transformation $x = \tilde{x} + \Pi_x x_{\mathrm{exo}}$ to (12): with

$$z_s^* = -\widetilde{K}_s\tilde{x} - \left(\widetilde{K}_s\Pi_x + \widetilde{R}^{-1}\left(\widetilde{B}^\top\Pi_v + R_w\right)\right)x_{\mathrm{exo}} \quad (20)$$

the stationary state $x_s(t) = \Pi_x x_{\mathrm{exo}}$ of (12) is given by

$$\Pi_x S = \left(\widetilde{A} - \widetilde{B}\widetilde{K}_s\right)\Pi_x - \widetilde{B}\widetilde{R}^{-1}\widetilde{B}^\top\Pi_v + \widetilde{B}_w. \quad (21)$$

Now examining the first term of (20) we formulate:

Optimal Control Problem B: Find $\tilde{z}(\cdot)$ that minimizes $\widetilde{J}(\tilde{z}(\cdot)) = \frac{1}{2}\int_{t_0}^{\infty} \tilde{x}^\top\widetilde{Q}\tilde{x} + 2\tilde{x}^\top\widetilde{S}\tilde{z} + \tilde{z}^\top\widetilde{R}\tilde{z}\ \mathrm{d}t$ with respect to the error dynamics $\dot{\tilde{x}} = \widetilde{A}\tilde{x} + \widetilde{B}\tilde{z}$ for $\tilde{x}(t_0) = x_0 - \Pi_x x_{\mathrm{exo},0}$. Then we obtain

Theorem 2: If Asmp. 2 holds, $\tilde{z}^* = -\widetilde{K}_s\tilde{x}$ with $\widetilde{K}_s = \widetilde{R}^{-1}(\widetilde{S}^\top + \widetilde{B}^\top P_s)$ as in Theorem 1 and $P_s \succeq 0$ given by ARE (16) is an optimal control of *Opt. Prob. B* for any $\tilde{x}(t_0)$.

Proof: The result is the well-known linear quadratic regulator, e.g. see [1] with adaptation for cross-terms. ■

In other words, $-\widetilde{K}_s\tilde{x}$ leads to an optimal transient $x \to \Pi_x x_{\mathrm{exo}}$. To continue, we formulate a constrained fixed end-point problem:

Optimal Control Problem C: Find $\Pi \in \mathbb{R}^{n\times n_{\mathrm{exo}}}$ and $\Gamma \in \mathbb{R}^{m-p\times n_{\mathrm{exo}}}$ such that $x(t) = \Pi x_{\mathrm{exo}}(t)$ and $z(t) = \Gamma x_{\mathrm{exo}}(t)$ $\forall t \in [t_0, t_1]$ solve *Opt. Prob. A* for $x(t_0) = \Pi x_{\mathrm{exo}}(t_0)$ with additional constraint $x(t_1) = \Pi x_{\mathrm{exo}}(t_1)$.

Now we ask, if $x = \Pi_x x_{\mathrm{exo}}$ is a unique optimal stationary solution of (12) regarding *Opt. Prob. C*. I.e., does the second term of (20) and $\Pi_x x_{\mathrm{exo}}$ uniquely solve *Opt. Prob. C*?

Theorem 3: For any $t_1 > t_0$ and any $x_{\mathrm{exo},0}$, conditions of Theorem 1 being satisfied, the optimal solution of *Opt. Prob. C* is uniquely given by $\Pi^* = \Pi_x$ and $\Gamma^* = -(\widetilde{K}_s\Pi_x + \widetilde{R}^{-1}(\widetilde{B}^\top\Pi_v + R_w))$ leading to a unique optimal stationary trajectory $x_s = \Pi_x x_{\mathrm{exo}}$ and optimal control $z_s^*|_{\tilde{x}=0} = \Gamma^* x_{\mathrm{exo}}$ which are obtained from (21) and (20), respectively.

Proof: Because of *Pontryagin's Minimum Principle*, it must hold $z = -\widetilde{R}^{-1}(\widetilde{S}^\top x + R_w x_{\mathrm{exo}} + \widetilde{B}^\top\lambda)$. Based on the stationarity constraint, it follows $\lambda = \Pi_\lambda x_{\mathrm{exo}}$, $\Pi_\lambda \in \mathbb{R}^{n\times n_{\mathrm{exo}}}$. Any optimal solution necessarily solves the Hamiltonian system [1]. Thus, for a stationary solution Πx_{exo} we find:

$$\begin{bmatrix}\Pi \\ \Pi_\lambda\end{bmatrix} S = \begin{bmatrix}\widetilde{A} - \widetilde{B}\widetilde{R}^{-1}\widetilde{S}^\top & -\widetilde{B}\widetilde{R}^{-1}\widetilde{B}^\top \\ -\widetilde{Q} + \widetilde{S}\widetilde{R}^{-1}\widetilde{S}^\top & -\left(\widetilde{A} - \widetilde{B}\widetilde{R}^{-1}\widetilde{S}^\top\right)^\top\end{bmatrix}\begin{bmatrix}\Pi \\ \Pi_\lambda\end{bmatrix} + \begin{bmatrix}B_w - \widetilde{B}\widetilde{R}^{-1}R_w \\ -S_w + \widetilde{S}\widetilde{R}^{-1}R_w\end{bmatrix} \quad (22)$$

with associated system matrix Θ being Hamiltonian. Consequently, we have $\sigma(\Theta) = \{\pm\lambda_i \mid i = 1, \ldots, n\}$ with $\{\lambda_i \mid i = 1, \ldots, n\} = \sigma(\widetilde{A} - \widetilde{B}\widetilde{R}^{-1}\widetilde{S}^\top - \widetilde{B}\widetilde{R}^{-1}\widetilde{B}^\top P_s)$ for $P_s \succeq 0$ given by (16), see [1]. It follows $\sigma(\Theta) \cap \sigma(S) = \emptyset$ and (22) has a unique solution Π^*, Π_λ^* and the boundary constraints are obviously met. Due to the fixed end-point the variation must satisfy $\delta x(t_1) = 0$ and optimality of Π^*, $\Gamma^* = -\widetilde{R}^{-1}(\widetilde{S}^\top\Pi^* + R_w + \widetilde{B}^\top\Pi_\lambda^*)$ can be verified by sufficiency exactly as in the proof of Lemma 3. Since the results of Theorem 1 also satisfy the necessary conditions, it follows $\Pi^* = \Pi_x$ and $\Gamma^* = z_s^*|_{\tilde{x}=0}$ by uniqueness. ■

As we see, (20) splits up into two parts: Firstly, a static feedback given by Theorem 2 which guarantees optimal transient behavior with asymptotic stability. Secondly, a static pre-filter which leads to a unique optimal stationary state trajectory in the sense of *Opt. Prob. C*. To be precise, solely the stationary trajectory $T_\eta\Pi_x x_{\mathrm{exo}}$ of the internal dynamics is optimized here. Finally, we are able to state the **main result** accounting for Problem I to IV:

Corollary 1: Suppose (2), Asmp. 1 and the assumptions of Theorem 1 hold. For any x_0 and $x_{\mathrm{exo},0}$, a stabilizing static decoupling control of system (1) with exogenous inputs by (4) exists which guarantees an optimal transient, cf. Theorem 2, to a unique optimal stationary trajectory, cf. Theorem 3, of internal dynamics. It is given by (10) with z by (18) based on ARE (16) and Sylvester equation (17).

Furthermore, as pointed out in Section IV-B, the resulting static control law is a valid approximation of the solution of *Opt. Prob. A* for a large final time t_1, as to expect, and a high rate of convergence α. The latter can be easily increased referring to Remark 2. This also allows us to extend the results in Coroll. 1 for unstable exo-systems, see Remark 3.

V. DISCUSSION AND SIMULATION RESULTS

This section will prove the applicability, practical value and, especially in comparison to other methods in the literature, the performance gain of our approach. Since the approximation of the optimal solution for finite final time is a main feature of our result, it is examined for each of the firstly discussed and secondly analyzed two examples:

In Example 1, we compare our results to others in the domain of static decoupling control under (2). Applying $z_+ = 0$ solves $\min_{z(t)} u(t)^\top R u(t)$ subject to (10) by means of the Moore-Penrose pseudoinverse in accordance to the use of DOF in [14], [18]. This equals the *control-allocation*-based solution in the unconstrained case and will indicate that methods like [12], [13] are not recommendable if $\operatorname{rank}(B) > p$. Furthermore, we consider z_∞ which minimizes the input energy gain γ, see Section I, by means of the *bounded real lemma* for the transfer matrix $G_{w\to u}(s)$ [22].

Example 2 deals with the optimization problem:

$$\min_{\Pi_{\text{tr}},\Gamma_{\text{tr}}} \text{trace}\left(\begin{bmatrix}\Pi_{\text{tr}}\\ \Gamma_{\text{tr}}\\ I\end{bmatrix}^\top \begin{bmatrix}\widetilde{Q} & \widetilde{S} & S_w\\ \widetilde{S}^\top & \widetilde{R} & R_w\\ S_w^\top & R_w^\top & Q_w\end{bmatrix}\begin{bmatrix}\Pi_{\text{tr}}\\ \Gamma_{\text{tr}}\\ I\end{bmatrix}\right) \tag{23}$$

with $\Pi_{\text{tr}} \in \mathbb{R}^{n\times n_{\text{exo}}}$ and $\Gamma_{\text{tr}} \in \mathbb{R}^{m-p\times n_{\text{exo}}}$ constrained to a stationary solution $x = \Pi_{\text{tr}} x_{\text{exo}}$, $z = \Gamma_{\text{tr}} x_{\text{exo}}$ of the system (12) with x_{exo} given by (4). This intends to minimize the integrand $L(x, z, t)$ of $J(z)$ in *Opt. Prob. A* in a stationary sense. This equals the objective in [20, Sec. V. B]. Furthermore, their suggested optimization for the DOF directly translates to (23) if applied to our framework with decoupling constraint. Optimization problem (23) equals the formulated problem in [16] which intended to minimize a quadratic cost functional similar to ours. In general, this approach based on [20] and [16] has a major drawback: the optimization (23) is “not invariant under change of coordinates” [16, p. 306] of the exo-system. Hence, the exo-system should satisfy the condition $x_{\text{exo}}^\top W x_{\text{exo}} = const.$ for some $W \succ 0$ and (4) should be modified by means of W, cf. [16]. Regarding the difficulty of finding W, it is easy to see that even for simple exo-systems of great interest, e.g. implementing polynomials in time, such a W may not exist. As in [20], this will be ignored in Example 2.

For the given purpose, we consider a real helicopter model *DRA Lynx ZD 559* linearized about longitudinal movement at velocity $v_x = 111$ km/h [19, p. 279] with dimensions $n = 9$, $m = 4 = \operatorname{rank}(B)$ and some unstable eigenvalues. The states are deviations of longitudinal/lateral velocities: x_1, x_5, descent rate: x_2 as well as pitch/roll/yaw angles: x_4, x_7, x_9 and turn rates: x_3, x_6, x_8.

Example 1: We want to perform an autonomous flight maneuver with decoupling of outputs $y = \begin{bmatrix}x_1 & x_5 & x_9\end{bmatrix}^\top$ and tracking of non-sinusoidal $w_1 = 0$, $w_2 = 18$ km/h, $w_3 = 10^\circ(\sin(^{0.2t}/_{\text{s}} + {}^{15\pi}/_{180}) - \sin(^{15\pi}/_{180})\mathrm{e}^{-0.1t/\text{s}})$ given by a 4th-order exo-system (4) with $O = \begin{bmatrix}I_{2,2} & 0 & e_2\end{bmatrix}$, $w^\top = \begin{bmatrix}0 & x_{\text{exo}}^\top O^\top\end{bmatrix}$ and $\sigma_{\text{exo}} = \{0, \pm j0.2, -0.1\}$. This leads to a slalom motion with sideslip. We have $\delta_1 = \delta_2 = 1$, $\delta_3 = 2$ and $\delta = 4$. It is desired $\sigma_1 = \sigma_2 = \{-1\}$, $\sigma_3 = \{-4, -4\}$. Applying Lemma 2 for $\alpha_j = 1\ \forall j$, gives decoupling K, F and stable internal dynamics; Theorem 1 can be applied. Since $\operatorname{rank}(B) > p = 3$ is satisfied, we have DOF $z \in \mathbb{R}$. We choose Q with element $q_{2,2} = 0.01$, zero else, in order to penalize vertical movement and $R = I$. Then, our method Coroll. 1 gives z_s^* in (18) by solving a 9×9 dim. ARE (16) and 9×4 dim. Sylvester equation (17).

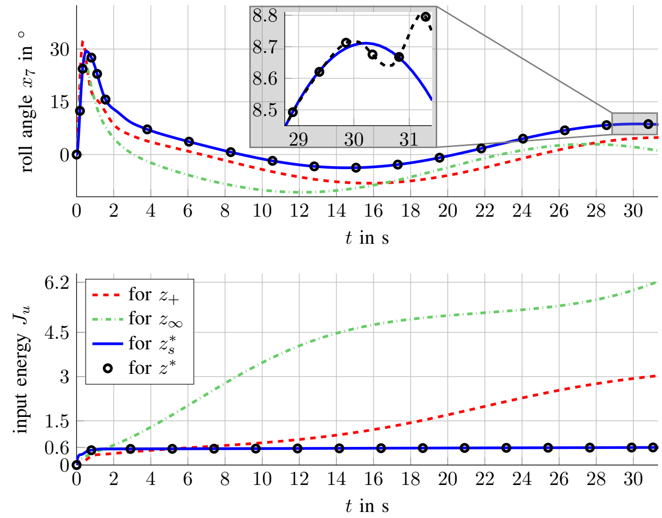

Fig. 1. In comparison to [14], [18] (similar to CA) and [22] (Example 1)

Simulation results for the initial values $x_1(0) = \Delta v_x = 18$ km/h, zero otherwise, over one period $t_1 = 31.4s$ are shown for roll angle x_7 and input energy $J_u(t) = \int_{t_0}^{t} u^\top R u\ \mathrm{d}t$ in Fig. 1. The performance in case of the *control-allocation*-based solution z_+ and z_∞, cf. [22], is unsatisfactory. This is caused by neglecting the internal dynamics in the first case and the inability to regard reference input spectra in the second. In contrast, the results regarding z_s^* are more than satisfying. As pointed out in Section IV-B, though $\alpha = 1.765$ only, our solution closely approximates the computationally very expensive optimal control z^* (54 scalars for each of sufficiently many time-steps need to be stored in advance plus interpolation of these during the application). Their costs nearly match with $J_s^* - J^* < 0.0002$ and the optimal trajectories, e.g. for x_7, vary only at the very end of the maneuver since $v_s^*(t_1) = \Pi_v x_{\text{exo}}(t_1) \neq v^*(t_1) = 0$ for example. In comparison, we have $J_u^* \approx J_{u,s}^* = 0.6$ versus $J_{u,+} = 3.03$, $J_{u,\infty} = 6.25$ and are saving input energy worth over 80% or 90%, respectively.

Considering the roll angle x_7^*, we can see that the circling motion is supported by leaning the helicopter into the turns. While averagely leaning to the right, only $e_7^\top \Pi_x x_{exo,1} = {}^{0.1305^\circ w_2}/_{\text{km/h}} = 2.35^\circ$ in x_7, also allows the main rotor thrust to account for the desired lateral velocity w_2. Furthermore, we manage ${}^{3.9\ \text{cm}}/_{\text{s}} < x_2 < {}^{4.7\ \text{cm}}/_{\text{s}}$ by choosing $q_{2,2} = 0.01$ and the helicopter maintains its altitude whereas $x_{2,+}, x_{2,\infty} < {}^{-10\ \text{m}}/_{\text{s}}$ which is unacceptable. Concluding, the optimal pre-filter in (20) leads to an intuitive motion of internal dynamics which also happens to be optimal. In contrast to the simplicity of our approach, determining these

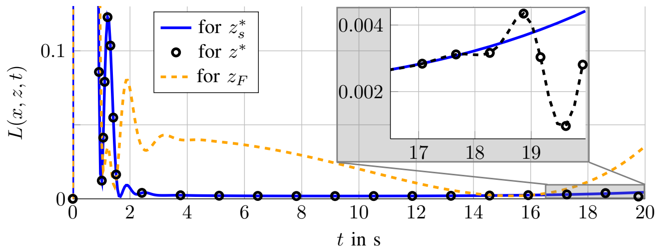

Fig. 2. In comparison to optimization in [16], [20] (Example 2)

trajectories explicitly and implementing a tracking controller for the square system, e.g. with additional output x_7, would be rather inconvenient though possible.

Example 2: A straight acceleration/deceleration maneuver with $v_x = 108$ km/h $\to$ 148 km/h $\to$ 108 km/h and yaw angle strictly $x_9 = 0$ is desired. The outputs are $y = \begin{bmatrix} x_1 & x_9 \end{bmatrix}^\top$ with $\delta_1 = 1$, $\delta_2 = 2$ and $\sigma_1 = \{-1\}$, $\sigma_2 = \{-4, -4\}$. The desired values follow: $w_1(t) = -0.4t^2 + 8t$ and $w_2(t) = 0$ on interval $[0, 20s]$. A 3rd-order exo-system for w_1 with $\sigma_{\text{exo}} = \{0, 0, 0\}$, $O = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $x_{\text{exo},0} = \begin{bmatrix} 0 & 8 & -0.8 \end{bmatrix}^\top$ is implemented. We choose $q_{2,2} = q_{5,5} = 0.1$, $q_{7,7} = 1$, zero else, in order to penalize vertical/lateral/rolling movement and $R = \text{diag}(100, 1, 1, 1)$ penalizing the main rotor collective such that a pitching motion which is cost-free $q_{3,3} = q_{4,4} = 0$ is preferred. For decoupling, Lemma 2 can be applied and our suggested optimal choice of the DOF Theorem 1 is feasible. Analogous to the proposed optimization in [16], [20], solving (23) gives $z_{\text{tr}} = -\widetilde{K}_s(x - \Pi_{\text{tr}}^* x_{\text{exo}}) + \Gamma_{\text{tr}}^* x_{\text{exo}}$ which also guarantees an optimal transient to $\Pi_{\text{tr}}^* x_{\text{exo}}$.

For $x_0 = 0$, the resulting intended objective $L(x, z, t)$ of (23) is shown in Fig. 2. While the results are comparable during transition $[0, 2\text{s}]$, the stationary performance apparently differs. Obviously, the suggested optimization [16], [20] fails to minimize $L(x, z, t)$ stationarily and leads to a worse performance. Regarding $J(z) = \int_{t_0}^{t_1} L(x, z, t)\, \text{d}t$, our result z_s^* approximates the computationally expensive optimal solution z^* closely by $J(z_s^*) = 100.12\%\ J(z^*)$. Based on the discussion in Section IV-B, the additional cost arises in the last $t_1 - t_v \approx 5/2\alpha + 5/\alpha = 3.8$s with $\alpha = 1.96$; see the zoom. In contrast, it results $J(z_{\text{tr}}) = 123.89\%\ J(z^*)$. The reason is that a W as discussed above does not exist.

Summarizing, our optimal use of the DOF clearly outdoes the proposed use in [16], [20] when adapted to our framework. In turn, our modular approach in Section IV could be adapted for a performance increase in [16], [20].

VI. Conclusion

Static decoupling control of linear *over-actuated* systems under time-varying references was considered. We have proven a static control law to be sufficient to account for an optimal transient behavior to a unique optimal stationary trajectory of internal dynamics. It can be simply calculated by solving an ARE and a Sylvester equation. Furthermore, the optimal stationary trajectory was derived explicitly which gave valuable insight in the optimized closed loop behavior. For a large final time and a high rate of convergence we have shown that it closely approximates the computationally expensive solution of the finite time optimal control problem. Finally, comprehensive simulations demonstrated very good performance of our approach in comparison with various others. This is a motivation for transferring the modular optimization approach to other frameworks of *over-actuated* systems. Furthermore, a generalization of the results in Section IV will be presented in a forthcoming paper.

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