Equilibria of continuous-time recurrent fuzzy systems

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Abstract

Unlike static fuzzy systems, recurrent fuzzy systems allow representing knowledge-based dynamic processes that can be stated in the form of "if . . . , then . . ." rules, making it possible to model systems that can only be described qualitatively. Further possible applications exist in the case where the dynamics of a system are quantitatively known, but only in certain mesh points. The interpolating character of the fuzzy system between the dynamics of the mesh points yields a complete dynamic model. Based on discrete-time recurrent fuzzy systems this article presents first steps towards the theory of continuous-time recurrent fuzzy systems and provides criteria for the investigation of the dynamics of this class of systems.

Keywords: Continuous-time recurrent fuzzy systems; Nonlinear dynamic systems; Interpolation; Equilibria; Stability; Fuzzy-Solow model

1. Introduction

In many cases mathematical models may be used to describe dynamic processes. A lot of these mathematical models lead to systems of differential equations or difference equations, generated from the knowledge of e.g. a physical process. Their dynamics can be investigated by applying analytical or numerical methods to these equations.

Modeling dynamics is a more difficult task in the case that describing a process is only qualitatively possible, often employing linguistic if–then rules. The quality of this form of description depends primarily on an accurate linguistic description of the dynamic behavior of the process, but less on exact values. The analysis of the dynamics can be accomplished by means of the linguistic if–then rules or by deriving a mathematical model from the qualitative model by application of fuzzy-logic which allows an analytical or numerical analysis of the dynamics.

In the discrete-time case different approaches which lead to fuzzy systems with inherent dynamics exist [4]: summarized, recurrent fuzzy systems and dynamic fuzzy systems have in common that they are rule based. Recurrent fuzzy systems [19,1–3,25,24,26] incorporate fuzzification, a standard inference method, and defuzzification. Investigating a state-graph representation of this kind of systems allows a dynamics analysis in most cases [3,25]. Dynamic fuzzy systems [37,38] map fuzzy sets on fuzzy sets using a more complex inference system, such that the information content and fuzziness of the states is known. By considering the mapping of the center and foot points of the fuzzy sets, a stability analysis can be accomplished [37,38]. In contrast to these two types of systems, iterated fuzzy sets [12,15,16,28] and a further type also called dynamic fuzzy systems [29,31,30,47,48,50] do not work rule

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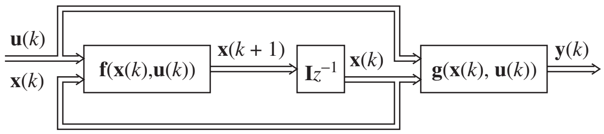

Fig. 1. Block schematic of a discrete-time recurrent fuzzy system, where *f* and *g* are fuzzy functions with fuzzification, inference, and defuzzification.

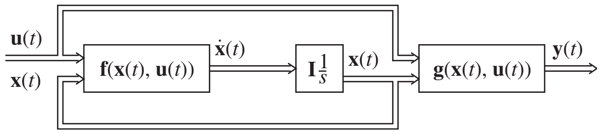

Fig. 2. Block schematic of a continuous-time recurrent fuzzy system, where *f* and *g* are fuzzy functions with fuzzification, inference, and defuzzification.

based but often use the extension principle of Zadeh to map fuzzy sets on fuzzy sets using functions, that may also be fuzzy.

Besides these discrete-time fuzzy systems, several continuous-time fuzzy systems with inherent dynamics exist: fuzzy differential equations [33,45,11,10] are derived from a generalization and extension of the iterated fuzzy sets. Continuous-time dynamic fuzzy systems [49] are a continuous-time approach for the second type of dynamic fuzzy systems. The dynamics analysis of these system types is complex, see e.g. [49,30,34,11,28,15,16]. In contrast to the systems above, fuzzy control systems are dynamic systems which consist of a fuzzy controller and a dynamic system, but the fuzzy controller itself does not have inherent dynamics.

Another continuous-time approach with inherent dynamics may be derived from the above-mentioned discrete-time recurrent fuzzy systems whose block schematic is illustrated in Fig. 1. These discrete-time recurrent fuzzy systems consist of a complete fuzzy system, $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k))$, covering fuzzification, inference, and defuzzification, whose inputs are the crisp state vector $\mathbf{x}(k)$ and the crisp input vector $\mathbf{u}(k)$ in the current time step k. Using these vectors the crisp state vector $\mathbf{x}(k+1)$ at the output of the fuzzy system is computed which is fed back to the input through a delay element. The output function $\mathbf{g}$ is also a complete fuzzy system to determine an output vector $\mathbf{y}$ for the input vector $\mathbf{u}$ and state vector $\mathbf{x}$.

Replacing the variables $\mathbf{x}(k)$, $\mathbf{u}(k)$, $\mathbf{y}(k)$, and the delay element in a discrete-time recurrent fuzzy system by the continuous-time variables $\mathbf{x}(t)$, $\mathbf{u}(t)$, $\mathbf{y}(t)$, and an integrator element, respectively, leads to the continuous-time recurrent fuzzy system whose structure is shown in Fig. 2. It also comprises a complete fuzzy system $\mathbf{f}$ whose inputs are the crisp state vector $\mathbf{x}(t)$ and the crisp input vector $\mathbf{u}(t)$ at a particular time t. For these inputs the crisp derivative of the state vector with respect to time $\dot{\mathbf{x}}(t)$ is computed at the output of the fuzzy system and fed back to the input of the fuzzy system through an integrator element. Even though both the discrete-time and the continuous-time recurrent fuzzy system have a similar structure, they require fundamentally different types of system theory [4,3].

Systems with such a kind of structure were already introduced in [5] for qualitative process modeling without denoting them continuous-time recurrent fuzzy systems. A few attempts to investigate the dynamics of these continuous-time systems were also accomplished in [5], however, a mathematical analysis of the dynamics has not been carried out yet. Furthermore, the analysis of the dynamics by means of the linguistic rules and the illustration of the close relationship between the structure of these systems and that of discrete-time recurrent fuzzy systems have remained open tasks. This article presents first steps of this not yet covered mathematical analysis.

Possible applications of continuous-time recurrent fuzzy systems arise among others in e.g. modeling dynamic processes in technical applications with only qualitatively known dynamics or in the field of economic theory, where the dynamics and the variation of quantities with respect to time are often linguistically describable. To describe both static and dynamic processes fuzzy logic is already applied in the technical field, e.g. in [1,2,26,37–39,8,41,42], and in the field of economic theory, e.g. in [13,44,14,41].

Various dynamic processes in the field of economic theory found in literature can also be described verbally, e.g. among others in [22,18,23,6,9,21,32]. Thus, we start off by demonstrating the principle of continuous-time recurrent fuzzy systems with a simple example in the field of economic theory: a model of the relationship between the wage rate and employment rate in an economic system described by the following rule types [22,18]:

(1) If the employment rate is above average and the wage rate is above average, then the change in employment rate is negative and the change in wage rate is positive.
(2) If the employment rate is below average and the wage rate is below average, then the change in employment rate is positive and the change in wage rate is negative.
(3) ....

Similar rules are additionally necessary to describe the dynamic relationship between employment rate and wage rate completely. The employment rate and wage rate in the if-part are regarded as states denoted $x_1(t)$ and $x_2(t)$ or combined into a state vector $\mathbf{x}(t)$. The then-part describes the change in employment rate and wage rate with respect to time denoted by the gradient $\dot{\mathbf{x}}$ of the state vector $\mathbf{x}$ leading to a linguistic dynamic model.

The evaluation of the linguistic rules is carried out using a complete fuzzy system $\mathbf{f}(\mathbf{x})$ which yields the structure of a continuous-time recurrent fuzzy system. In this simple manner one may derive an easy, comprehensible dynamic model of the relationship between wage rate and employment rate. Compared to this method, the direct derivation of a mathematical description of the above depicted dynamic process, e.g. with differential equations, is more difficult.

In contrast to the above described approach, the methods of Takagi and Sugeno [43,42] use local linear sub-models to describe the dynamics of the process. The linear models are either difference equations or differential equations which are weighted by fuzzy rules. The dynamics are fully specified by the equations, but not by a linguistic description, appearing in the conclusion part of the rules involved. Both discrete-time and continuous-time recurrent fuzzy systems directly describe the dynamics of the process by means of rules, such that the Takagi and Sugeno type methods cannot be applied.

In the following sections, it becomes apparent how a mathematical model of a system may be obtained from the linguistic representation of a continuous-time recurrent fuzzy system using fuzzy logic. This results in a system of nonlinear differential equations. The linguistic if–then rules already show similarity to differential equations, so that we also call the if–then rules of continuous-time recurrent fuzzy systems linguistic differential equations.

In Section 2, continuous-time recurrent fuzzy systems and their mathematical representation are defined in detail. Equilibrium points of this kind of systems are investigated in Section 3. Section 4 elucidates the procedure for modeling dynamic processes in detail using continuous-time recurrent fuzzy systems. Subsequently, we illustrate the modeling process and the application of the stability criteria by means of an example in Section 5.

2. Definition of continuous-time recurrent fuzzy systems

In the first subsection, we start off by introducing a formal description of continuous-time recurrent fuzzy systems which is closely related to that of discrete-time recurrent fuzzy systems [3,25,24]. In the second subsection, we define the employed fuzzy systems in detail.

2.1. Formal representation of rules

We give a formal description for continuous-time recurrent fuzzy systems which we illustrate by employing the simplified model out of the field of economic theory from the previous section. Analogously to the states of discrete-time recurrent fuzzy systems, the employment rate and wage rate in the example are said to be the linguistic states, $x_1(t)$ and $x_2(t)$, respectively. The adjectives "below average" and "above average" that specify the grade of the employment rate and wage rate are said to be the linguistic values, $L_{j_1}^{x_1}$ and $L_{j_2}^{x_2}$ with $j_1, j_2 \in \{1; 2\}$, of the linguistic states, x_1 and x_2, respectively as already mentioned in the previous section. E.g. the component $j_1 = 1$ indicates the linguistic value $L_1^{x_1}$ = "below average" and $j_1 = 2$ indicates the linguistic value $L_2^{x_1}$ = "above average" of the linguistic state x_1. In the case that a linguistic state, x_i, can take on more than two values, i.e. n values, we define $j_i \in \{1, 2, \ldots, n\}$.

The then-part of a rule linguistically describes the variation of the states with respect to time. We define:

Definition 1 (*Definition of "linguistic derivative"*). The term $\dot{x}_i$, that linguistically describes the change in the linguistic state x_i with respect to time, is said to be the linguistic derivative of x_i.

Returning to the example, the variation of the wage rate described in the consequent of a rule corresponds to the linguistic derivative "change in wage rate" of the linguistic state "wage rate." Formally, the linguistic values that describe the degree of change of a linguistic state x_i are denoted $L^{\dot{x}_i}_{w_i(\mathbf{j})}$. The index $w_i(\mathbf{j})$ indicates that each rule defines a mapping of the index vector $\mathbf{j} = (j_1, \ldots, j_n)^{\mathrm{T}}$, onto the components, w_i, of an index vector $\mathbf{w} = (w_1, \ldots, w_n)^{\mathrm{T}}$. Linguistic derivatives can be positive, e.g. indicated by the linguistic value $L^{\dot{x}_i}_{w_i(\mathbf{j})}$ = "positive," which yields $\dot{x}_i(t) > 0$ or negative, e.g. indicated by the linguistic value $L^{\dot{x}_i}_{w_i(\mathbf{j})}$ = "negative," so that $\dot{x}_i(t) < 0$. They can also be zero with $\dot{x}_i(t) \approx 0$, e.g. indicated by the expression "is zero" [5].

By employing the notation introduced above the rules have the following formal form [5]:

$$\text{If } x_1(t) = L^{x_1}_{j_1} \quad \text{and} \quad x_2(t) = L^{x_2}_{j_2}, \qquad \text{then } \dot{x}_1(t) = L^{\dot{x}_1}_{w_1(\mathbf{j})} \quad \text{and} \quad \dot{x}_2(t) = L^{\dot{x}_2}_{w_2(\mathbf{j})}. \tag{1}$$

In general, a dynamic system also involves inputs, $u_p(t)$. The finite number of linguistic values of a linguistic input, $u_p(t)$, is named $L^{u_p}_{q_p}$ with $q_p \in \{1, 2, \ldots, m\}$. Incorporating this, we obtain rules of the form:

$$\begin{aligned} &\text{If} \quad x_1(t) = L^{x_1}_{j_1} \text{ and} \ldots \text{and } x_n(t) = L^{x_n}_{j_n},\\ &\text{and} \quad u_1(t) = L^{u_1}_{q_1} \text{ and} \ldots \text{and } u_m(t) = L^{u_m}_{q_m},\\ &\text{then} \quad \dot{x}_1(t) = L^{\dot{x}_1}_{w_1(\mathbf{j},\mathbf{q})} \text{ and} \ldots \text{and } \dot{x}_n(t) = L^{\dot{x}_n}_{w_n(\mathbf{j},\mathbf{q})}. \end{aligned} \tag{2}$$

In this case, each rule defines a mapping of both the index vector $\mathbf{j} = (j_1, \ldots, j_n)^{\mathrm{T}}$ and the index vector $\mathbf{q} = (q_1, \ldots, q_m)^{\mathrm{T}}$ onto the component w_i of an index vector $\mathbf{w} = (w_1, \ldots, w_n)^{\mathrm{T}}$ which is indicated by using $L^{\dot{x}_i}_{w_i(\mathbf{j},\mathbf{q})}$ to denote the linguistic values of the linguistic derivative, $\dot{x}_i$. In the following, we introduce the linguistic state vector, $\mathbf{x} \in \mathbb{R}^n$, and linguistic input vector, $\mathbf{u} \in \mathbb{R}^m$, with the linguistic vectors, $\mathbf{L}^{\mathbf{x}}_{\mathbf{j}}$ and $\mathbf{L}^{\mathbf{u}}_{\mathbf{q}}$, respectively, and the linguistic derivative vector, $\dot{\mathbf{x}} \in \mathbb{R}^n$, with the associated linguistic gradients, $\mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j},\mathbf{q})}$.

Definition 2 (*Definition of "linguistic vector"*). The vector $\mathbf{L}^{\mathbf{x}}_{\mathbf{j}} = (L^{x_1}_{j_1}, L^{x_2}_{j_2}, \ldots, L^{x_n}_{j_n})^{\mathrm{T}}$, where the element $i \in \{1, \ldots, n\}$ is a linguistic value $L^{x_i}_{j_i}$ of the linguistic variable x_i, is said to be a linguistic vector.

The linguistic vector $\mathbf{L}^{\mathbf{u}}_{\mathbf{q}} = (L^{u_1}_{q_1}, L^{u_2}_{q_2}, \ldots, L^{u_m}_{q_m})^{\mathrm{T}}$ is defined similarly.

Definition 3 (*Definition of "linguistic gradient"*). The vector $\mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{w}} = (L^{\dot{x}_1}_{w_1}, L^{\dot{x}_2}_{w_2}, \ldots, L^{\dot{x}_n}_{w_n})^{\mathrm{T}}$, where the element $i \in \{1, \ldots, n\}$ is a linguistic value $L^{\dot{x}_i}_{w_i}$ of the linguistic derivative $\dot{x}_i$, is said to be a linguistic gradient.

The linguistic vectors, $\mathbf{L}^{\mathbf{x}}_{\mathbf{j}}$, $\mathbf{L}^{\mathbf{u}}_{\mathbf{q}}$, and the linguistic gradient, $\mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j},\mathbf{q})}$, represent the "and" correlation of their respective components, e.g. the expression $\mathbf{x} = \mathbf{L}^{\mathbf{x}}_{\mathbf{j}} = (L^{x_1}_{j_1}, L^{x_2}_{j_2}, L^{x_3}_{j_3}, \ldots)^{\mathrm{T}}$ is a short form for " $x_1 = L^{x_1}_{j_1}$ and $x_2 = L^{x_2}_{j_2}$ and $x_3 = L^{x_3}_{j_3}$ and ... ."

Thus, the rules (2) may be written in the short form,

$$\text{If } \mathbf{x}(t) = \mathbf{L}^{\mathbf{x}}_{\mathbf{j}} \quad \text{and} \quad \mathbf{u}(t) = \mathbf{L}^{\mathbf{u}}_{\mathbf{q}}, \qquad \text{then } \dot{\mathbf{x}}(t) = \mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j},\mathbf{q})}. \tag{3}$$

Instead of listing all rules of a rule base, the representation as a rule matrix is a concise alternative. The rule matrix for the above example describing the wage rate and employment rate of an economic system is shown in Table 1.

This linguistic representation of a continuous-time recurrent fuzzy system may be formalized mathematically by employing a differential equation $\mathbf{f}$ and a static output function $\mathbf{g}$ defined by the equations,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \tag{4}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)). \tag{5}$$

As already mentioned, the functions $\mathbf{f}$ and $\mathbf{g}$ are fuzzy functions covering fuzzification, inference, and defuzzification. They allow mathematical processing and investigation of the linguistic rules.

In the following subsection, we define the configuration of the vector fuzzy function, $\mathbf{f}(\mathbf{x}, \mathbf{u})$. The vector fuzzy system, $\mathbf{g}(\mathbf{x}, \mathbf{u})$, is similarly configured, but does not have impact on the dynamics of the whole system, since the

Table 1
Rule base showing the simplified model for the change in wage rate $L^{x_1}_{j_1}$ and employment rate $L^{x_2}_{j_2}$ in an economic system

		Wage rate $L^{x_1}_{j_1}$		Wage rate $L^{x_1}_{j_1}$	
		ba	aa	ba	aa
Employment	aa	Positive	Positive	Positive	Negative
Rate $L^{x_2}_{j_2}$	ba	Negative	Negative	Positive	Negative
		(a) Change in wage rate $L^{\dot{x}_1}_{w_1(\mathbf{j})}$		(b) Change in employment rate $L^{\dot{x}_2}_{w_2(\mathbf{j})}$	

ba = below average, aa = above average.

function, $\mathbf{g}(\mathbf{x}, \mathbf{u})$, is configured as a static fuzzy system. Thus, we neglect this function henceforth, even though both $\mathbf{f}$ and $\mathbf{g}$ together define a continuous-time recurrent fuzzy system.

2.2. Definition of the fuzzy system

The rule base of the fuzzy system, $\mathbf{f}$, should be both free of contradictions and complete. Fuzzification of the if-part involves assigning a membership function, $\mu^{x_i}_{j_i}(x_i)$, to every linguistic value, $L^{x_i}_{j_i}$, of each state, x_i, on an interval $X_i \subseteq \mathbb{R}$. The membership functions $\mu^{u_p}_{q_p}(u_p)$ are defined similarly. Singletons, $s^{\dot{x}_i}_{w_i(\mathbf{j},\mathbf{q})}$, are employed as membership functions for the linguistic values, $L^{\dot{x}_i}_{w_i(\mathbf{j},\mathbf{q})}$, of the linguistic derivatives, $\dot{x}_i$. Additionally, we define the so-called core positions and the core position derivative:

Definition 4 (*Definition of "core position"*). A value $s^{x_i}_{j_i}$ for every linguistic value, $L^{x_i}_{j_i}$, of a linguistic variable x_i, where the according membership function, $\mu^{x_i}_{j_i}(x_i)$, takes on its maximum value, is said to be a core position.

The core positions $s^{u_p}_{q_p}$ are defined similarly.

Definition 5 (*Definition of "core position derivative"*). A singleton, $s^{\dot{x}_i}_{w_i}$, for the linguistic value, $L^{\dot{x}_i}_{w_i}$, of a linguistic derivative, $\dot{x}_i$, is said to be a core position derivative.

By combining the core positions and core position derivatives as elements of different vectors, we define the core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$ and $\mathbf{s}^{\mathbf{u}}_{\mathbf{q}}$, and core position gradient, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j},\mathbf{q})}$, respectively.

Definition 6 (*Definition of "core position vector"*). The vector $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}} = (s^{x_1}_{j_1}, s^{x_2}_{j_2}, \ldots, s^{x_n}_{j_n})^{\mathrm{T}}$, where the element $i \in \{1, \ldots, n\}$ is the core position $s^{x_i}_{j_i}$ for the linguistic value $L^{x_i}_{j_i}$ of the linguistic variable x_i, is said to be a core position vector.

The core position vector $\mathbf{s}^{\mathbf{u}}_{\mathbf{q}} = (s^{u_1}_{q_1}, s^{u_2}_{q_2}, \ldots, s^{u_m}_{q_m})^{\mathrm{T}}$ is defined similarly.

Definition 7 (*Definition of "core position gradient"*). The vector $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}} = (s^{\dot{x}_1}_{w_1}, s^{\dot{x}_2}_{w_2}, \ldots, s^{\dot{x}_n}_{w_n})^{\mathrm{T}}$, where the element $i \in \{1, \ldots, n\}$ is the core position derivative $s^{\dot{x}_i}_{w_i}$ for the linguistic value $L^{\dot{x}_i}_{w_i}$ of the linguistic derivative $\dot{x}_i$, is said to be a core position gradient.

The core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}} \in X = \mathbb{R}^n$ and $\mathbf{s}^{\mathbf{u}}_{\mathbf{q}} \in U = \mathbb{R}^m$, and the combinations thereof, form a lattice in the state space, X, the input space, U, and the space, $X \times U$, respectively. There is only one active rule in any core position vector, $(\mathbf{x}, \mathbf{u}) = (\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}, \mathbf{s}^{\mathbf{u}}_{\mathbf{q}})$, such that only one core position gradient, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j},\mathbf{q})}$, is associated to each combination $(\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}, \mathbf{s}^{\mathbf{u}}_{\mathbf{q}})$. The core position gradient, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j},\mathbf{q})}$, defines the change in the state value with respect to time in the corresponding lattice point, $(\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}, \mathbf{s}^{\mathbf{u}}_{\mathbf{q}})$, i.e. in which direction a trajectory points in this lattice point and how fast it passes through this lattice point.

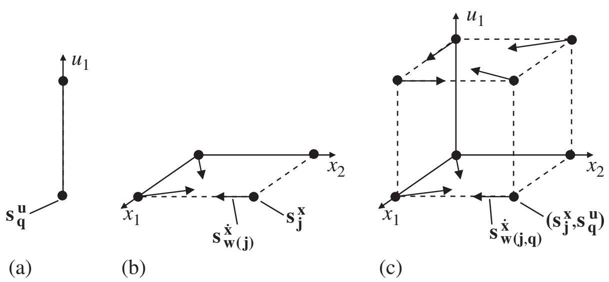

Fig. 3. The core position vectors form a lattice in the spaces (a) U, (b) X und (c) $X \times U$. The core position gradients are represented as arrows.

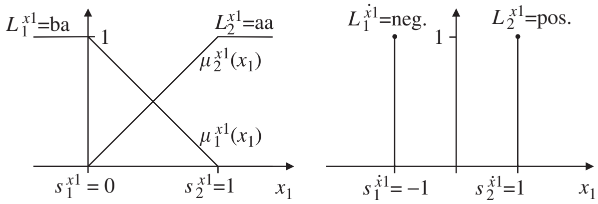

Fig. 4. Linguistic values, $L_{j_i}^{x_i}$, and derivatives, $L_{w_i(\mathbf{j})}^{\dot{x}_i}$, respectively, membership functions, $\mu_{j_i}^{x_i}(x_i)$, core positions, $s_{j_i}^{x_i}$, and core position derivatives, $s_{w_i(\mathbf{j})}^{\dot{x}_i}$, for the component x_1 of the economic theory example. aa = above average, ba = below average, neg. = negative, pos. = positive.

Fig. 3 elucidates this representation. Considering a state vector, $\mathbf{x}$, that does not lie on a lattice point, the fuzzy system interpolates the associated gradient, $\dot{\mathbf{x}}$, between the gradients of the lattice points.

The membership functions, $\mu_{j_i}^{x_i}(x_i)$, must satisfy the following conditions according to [3,25,24]:

(1) Delimitation: $\mu_{j_i}^{x_i}(x_i) \in [0, 1]$ for all $x_i \in X_i$,

(2) Convexity: $\begin{cases} \mu_{j_i}^{x_i}(x_i) & \text{monotonically increases for all } x_i < s_{j_i}^{x_i} \\ \mu_{j_i}^{x_i}(x_i) & \text{monotonically decreases for all } x_i > s_{j_i}^{x_i} \end{cases}$,

(3) Partition: $\sum_{j_i} \mu_{j_i}^{x_i}(x_i) = 1$ for all $x_i \in X_i$, and $\mu_{j_i}^{x_i}(s_{j_i}^{x_i}) = 1$ and $\mu_{j_i}^{x_i}(s_{l_i}^{x_i}) = 0$ for $j_i \neq l_i$,

(4) Continuity: $\mu_{j_i}^{x_i}(x_i) \in [0, 1]$ is continuous in X_i.

These conditions should also hold for each membership function, $\mu_{q_p}^{u_p}(u_p)$, and their core positions, $s_{q_p}^{u_p}$, for each linguistic value, $L_{q_p}^{u_p}$.

The singletons, $s_{w_i(\mathbf{j},\mathbf{q})}^{\dot{x}_i}$, to fuzzify the output values, i.e. the linguistic value, $L_{w_i(\mathbf{j},\mathbf{q})}^{\dot{x}_i}$, may be chosen with arbitrary distances.

Employing again the example, i.e. the simplified economic model, Fig. 4 illustrates a possible choice for the membership functions, $\mu_{j_i}^{x_i}(x_i)$, the core positions, $s_{j_i}^{x_i}$, and the core position derivatives $s_{w_i(\mathbf{j})}^{\dot{x}_i}$ for the state variable x_1.

Algebraic multiplication is employed as both the aggregation operator and the implication operator, and summation is employed as the accumulation operator. Defuzzification is based on the center of singletons defuzzification (CoS). The choice of these operators leads to a continuous and differentiable fuzzy function, $\mathbf{f}(\mathbf{x}, \mathbf{u})$.

Under consideration of the conditions above and according to [3,25,24], we obtain the fuzzy function, $\mathbf{f}(\mathbf{x}, \mathbf{u})$, with the following analytical form:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \sum_{\mathbf{j},\mathbf{q}} \mathbf{s}_{\mathbf{w}(\mathbf{j},\mathbf{q})}^{\dot{\mathbf{x}}} \prod_{i=1}^{n} \mu_{j_i}^{x_i}(x_i) \prod_{p=1}^{m} \mu_{q_p}^{u_p}(u_p). \tag{6}$$

In the following section, we shall treat the investigation of equilibrium points of continuous-time recurrent fuzzy systems. Only autonomous systems, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, are investigated such that the index vector $\mathbf{q}$ is not incorporated in the following. In principle, the following investigations may also be similarly applied to non-autonomous systems.

3. Equilibria

Equilibrium points of continuous-time recurrent fuzzy systems may be divided into two fundamental different classes. Equilibrium points of the first category are located exactly on core position vectors, only one rule of the rule base being active. The second category comprises equilibrium points other than core position vectors which have several active rules.

3.1. Stability criteria for equilibrium points on core position vectors

Equilibrium points located exactly on core positions can be immediately read off from the rule base. Since there is always only one active rule on a core position vector, an equilibrium point exists if the consequent of the active rule does not yield a change in the state in the premise on this core position vector. The following theorem provides a formal description for the localization of this kind of equilibrium points:

Theorem 1. *If a core position vector*, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, *of an autonomous*, *continuous-time recurrent fuzzy system*, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, *is associated with the core position gradient*, $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}} = \mathbf{0}$, *then the core position vector*, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, *is an equilibrium point of the system*, $\mathbf{f}$.

Proof. For a core position vector, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, associated with the core position gradient $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}} = \mathbf{0}$, we have with Eq. (6) that

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x} = \mathbf{s}_{\mathbf{j}}^{\mathbf{x}}) = \sum_{\mathbf{j}} \mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}} \prod_{i=1}^{n} \mu_{j_i}^{x_i}(s_{j_i}^{x_i}) = \mathbf{0} \cdot 1 = \mathbf{0}, \tag{7}$$

since there is only one active rule at the considered core position vector. □

Without loss of generality, the equilibrium point is assumed to be located at $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} = \mathbf{0}$, since any other core position vector $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} \neq \mathbf{0}$ can be transformed accordingly.

Before starting off with the stability analysis of this kind of equilibrium points, we will introduce further definitions to simplify matters in the following. The definitions below are introduced in [3,24] for discrete-time recurrent fuzzy systems and are also applicable to continuous-time recurrent fuzzy systems.

Definition 8 (*Correlation of core position vectors and other vectors*). A core position vector, $(\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}, \mathbf{s}_{\mathbf{q}}^{\mathbf{u}})$, and another vector, $(\mathbf{x}, \mathbf{u})$, are said to be correlated iff there is only one, single, core position, i.e. $s_{j_i}^{x_i}$ or $s_{q_p}^{u_p}$, respectively, that falls within every closed interval bounded by the values of the components of the vectors $(\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}, \mathbf{s}_{\mathbf{q}}^{\mathbf{u}})$ and $(\mathbf{x}, \mathbf{u})$.

Definition 9 (*Definition of "adjacent"*). Two core position vectors are said to be adjacent iff only one of their components differ and no further core positions fall within the range bounded by the two values of the differing component. Two linguistic vectors, $\mathbf{L}_{\mathbf{j}}^{\mathbf{x}}$ (or $\mathbf{L}_{\mathbf{q}}^{\mathbf{u}}$), are said to be adjacent iff their core position vectors are adjacent. Two rules are said to be adjacent iff their linguistic state/input vectors, $(\mathbf{L}_{\mathbf{j}}^{\mathbf{x}}, \mathbf{L}_{\mathbf{q}}^{\mathbf{u}})$, are adjacent with respect to their premises.

Definition 10 (*Definition of "elementary hypersquare"*). The set of core position vectors correlated to a vector, $(\mathbf{x}, \mathbf{u})$, form the corners of a hypersquare. This hypersquare is said to be the elementary hypersquare of the vector $(\mathbf{x}, \mathbf{u})$.

Similar to standardized discrete-time recurrent fuzzy systems in [25] we define standardized continuous-time recurrent fuzzy systems with the following slightly different definition:

Definition 11 (*Definition of "standardized" continuous-time recurrent fuzzy systems*). A continuous-time recurrent fuzzy system will be termed "standardized" if equidistant triangular functions and ramp functions at the boundary respectively are employed as membership functions in fuzzifying its state variables.

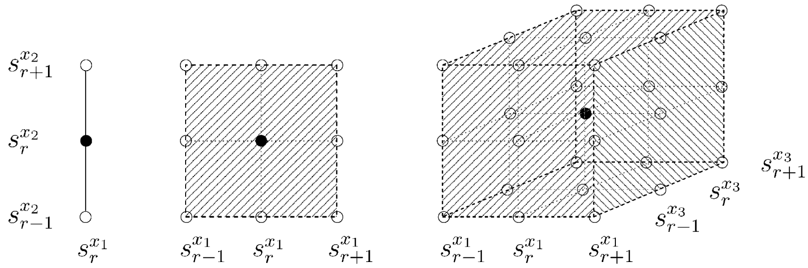

Fig. 5. Illustration of the elementary neighborhood (hatched) of a core position vector "$\bullet$" for one-, two-, and three-dimensional systems.

Choosing equidistant triangular membership functions, $\mu^{x_i}_{j_i}(x_i)$, i.e. equidistant core positions, $s^{x_i}_{j_i}$, and triangular membership functions, $\mu^{x_i}_{j_i}(x_i)$, does not constitute a severe restriction, since continuous-time recurrent fuzzy systems having non-equidistant core positions can be transformed into a continuous-time recurrent fuzzy system having equidistant core positions. Without loss of generality, it is furthermore possible to normalize a continuous-time recurrent fuzzy system, i.e. all edges of each of its hypersquares have the length one. Consequently, we introduce the following definition:

Definition 12 (*Definition of "normalized" continuous-time recurrent fuzzy systems*). A standardized continuous-time recurrent fuzzy system will be termed "normalized" iff all its elementary hypersquares have only edges of length $\Delta s_i = 1$ for all coordinates $i \in \{1, \ldots, n\}$.

Any standardized continuous-time recurrent fuzzy system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, may be transformed into a normalized continuous-time recurrent fuzzy system, $\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{f}}(\tilde{\mathbf{x}})$, using

$$\tilde{x}_i = \frac{x_i}{\Delta s_i}, \quad \dot{\tilde{x}}_i = \frac{\dot{x}_i}{\Delta s_i}, \tag{8}$$

where $\Delta s_i = s^{x_i}_{r+1} - s^{x_i}_{r}$ is the distance of two adjacent core positions, $s^{x_i}_{r}$ and $s^{x_i}_{r+1}$, in the i-th coordinate of the standardized system. Note, all core positions in the i-th coordinate are equally distributed, since the system is standardized, such that any two adjacent core positions, $s^{x_i}_{r}$ and $s^{x_i}_{r+1}$, may be used to compute Δs_i. By using Eq. (8) we obtain for an autonomous system $\dot{\tilde{x}}_i = \tilde{f}_i(\tilde{\mathbf{x}}) = (1/\Delta s_i) \sum_{\mathbf{j}} s^{\dot{x}_i}_{w_i(\mathbf{j})} \prod_k \mu^{x_k}_{j_k}(\Delta s_i \tilde{x}_i)$. The case $\Delta s_i < 1$ is an expansion of all hypersquares of the standardized system and the gradients of their vertices in the i-th coordinate. The case $\Delta s_i > 1$ is a shrinking. Both cases lead to normalized hypersquares and therefore to a normalized system. The linear transformation above does not change the system dynamics.

Consider a core position vector, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j0}}$. The rules of the hypersquares that have $\mathbf{s}^{\mathbf{x}}_{\mathbf{j0}}$ as vertex determine solely the dynamics of the system in the neighborhood of the core position vector, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j0}}$. Therefore, the union set of these elementary hypersquares is said to be the elementary neighborhood of the core position vector $\mathbf{s}^{\mathbf{x}}_{\mathbf{j0}}$. It is defined as follows:

Definition 13 (*Definition of the "elementary neighborhood" of a core position vector*). The set of vectors, $\mathbf{x}$ with $x_i \in \,]s^{x_i}_{r-1}, s^{x_i}_{r+1}[$, for all elements i form the elementary neighborhood, U_e, of a core position vector with $s^{x_i}_{r}$ for all elements i.

Note, the elementary neighborhood is an open set. Each core position vector, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, is sole member of its own elementary neighborhood and does not belong to the elementary neighborhood of another core position vector. In order to illustrate the elementary neighborhood of a core position vector, $\mathbf{s}^{\mathbf{x}}_{r}$, Fig. 5 depicts the one-, two-, and three-dimensional case.

The definition above provides in the case of an equilibrium point $\mathbf{x}_e = \mathbf{s}^{\mathbf{x}}_{\mathbf{j0}}$ lying precisely on a core position vector, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j0}}$, that the dynamics of the system have to be considered only within its elementary neighborhood to determine whether the considered equilibrium point is stable or unstable. Thus, the stability of the equilibrium point results from the core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, with the associated core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, on the boundary, δU_e, of the elementary neighborhood, U_e. Using this property the following theorem gives a criterion to analyze the stability of normalized continuous-time recurrent fuzzy systems having an equilibrium point, $\mathbf{x}_e$ with $\dot{\mathbf{x}}_e = \mathbf{0}$, that lies on the core position vector $\mathbf{s}_{\mathbf{j0}}^{\mathbf{x}} = \mathbf{0}$.

Theorem 2 (*Stability in the sense of Lyapunov*). *Let the core position vector $\mathbf{s}_{\mathbf{j_0}}^{\mathbf{x}} = \mathbf{0}$ be an equilibrium point of the normalized continuous-time recurrent fuzzy system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. This equilibrium point is stable in the sense of Lyapunov if all $3^n - 1$ core position gradients, $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}}$, whose associated core position vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, lie on the boundary of the elementary neighborhood of $\mathbf{x}_e$, satisfy the condition*

$$\sum_{i\in I^+(\mathbf{j})} |s_{w_i(\mathbf{j})}^{\dot{x}_i}| \leqslant \sum_{i\in I^-(\mathbf{j})} |s_{w_i(\mathbf{j})}^{\dot{x}_i}|,$$

where $I^+(\mathbf{j}) = \{i|s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} \geqslant 0\}$ and $I^-(\mathbf{j}) = \{i|s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} < 0\}$.

Note, each index vector, $\mathbf{j} = (j_1, \ldots, j_n)^{\mathrm{T}}$, describes one core position vector, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, where j_i indicates the linguistic value of the i-th element.

Proof. The proof is based on the stability theorem of Lyapunov for non-differentiable functions, presented in Theorem 7 in Appendix A.1. We choose the 1-norm of the state vector, $\mathbf{x}$, as a potential Lyapunov function,

$$v(\mathbf{x}) = \|\mathbf{x}\|_1 = \sum_{i\in I} |x_i| \quad \text{with } I = \{1, \ldots, n\}, \tag{9}$$

which obviously satisfies Conditions (1) and (2) of Theorem 7.

Subsequently, we verify that Condition (3) of Theorem 7,

$$\bar{D}^+ v(\mathbf{x}(t)) = \limsup_{\Delta t\to 0^+} \frac{\sum_{i\in I}[|x_i + \Delta t\dot{x}_i| - |x_i|]}{\Delta t} \leqslant 0, \tag{10}$$

is satisfied within the region $U_0 = \{\mathbf{x}|v(\mathbf{x})\leqslant 1\} = \{\mathbf{x}|\sum_{i\in I}|x_i|\leqslant 1\}$, where $\bar{D}^+ v(\mathbf{x}(t))$ is the upper Dini-derivative of v.

We start off by determining an appropriate representation of the differential equations of the recurrent fuzzy system. For this purpose, the n elements $s_{j_i}^{x_i}$ with the associated derivatives $s_{w_i(\mathbf{j})}^{\dot{x}_i}$ of the $3^n - 1$ core position vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, lying on the boundary, δU_e, of the elementary neighborhood of the equilibrium point, $\mathbf{x}_e$, are considered. These core positions vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, in fact their indices, $\mathbf{j}$, may be separated into the set $J^+(i) = \{\mathbf{j}|s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} \geqslant 0\}$ and the set $J^-(i) = \{\mathbf{j}|s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} < 0\}$ depending on i. Note, $J^-(i) \neq \emptyset$ holds for all i, if all core position vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, and their gradients, $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}}$, satisfy the condition of Theorem 2. Furthermore, we define the set $J = J^-(i) \cup J^+(i)$. Note, that J is independent of the index i and includes the index vectors, $\mathbf{j}$, of all $3^n - 1$ core position vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, lying on δU_e.

Then, we obtain for the element x_i of the state vector, $\mathbf{x}$, the derivative

$$\dot{x}_i = \sum_{\mathbf{j}\in J} s_{w_i(\mathbf{j})}^{\dot{x}_i} \prod_{k=1}^{n} \mu_{j_k}^{x_k}(x_k) = \sum_{\mathbf{j}\in J^-(i)} s_{w_i(\mathbf{j})}^{\dot{x}_i} \prod_{k=1}^{n} \mu_{j_k}^{x_k}(x_k) + \sum_{\mathbf{j}\in J^+(i)} s_{w_i(\mathbf{j})}^{\dot{x}_i} \prod_{k=1}^{n} \mu_{j_k}^{x_k}(x_k). \tag{11}$$

All core position vectors with an index vector $\mathbf{j} \in J^-(i)$ satisfy the condition $s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} < 0$, such that we get $s_{w_i(\mathbf{j})}^{\dot{x}_i} < 0$ in the case of $s_{j_i}^{x_i} > 0$ and $s_{w_i(\mathbf{j})}^{\dot{x}_i} > 0$ in the case of $s_{j_i}^{x_i} < 0$. Additionally, we have that $\prod_k \mu_{j_k}^{x_k}(x_k)$ is positive or zero. However, all $\prod_k \mu_{j_k}^{x_k}(x_k)$ never take the value zero simultaneously in the sum over all $\mathbf{j} \in J^-(i)$, i.e. a state vector $\mathbf{x}$ with a positive element $0 < x_i \leqslant s_{j_i}^{x_i}$ has a membership $\prod_k \mu_{j_k}^{x_k}(x_k) > 0$ to core position vectors with the component $s_{j_i}^{x_i} > 0$ and a membership $\prod_k \mu_{j_k}^{x_k}(x_k) = 0$ to the core position vectors with the component $s_{j_i}^{x_i} < 0$. This yields in the sum over all $\mathbf{j} \in J^-(i)$ the inequality $-\infty < \sum_{\mathbf{j}\in J^-(i)} s_{w_i(\mathbf{j})}^{\dot{x}_i} \prod_k \mu_{jk}^{x_k}(x_k) < 0$. Analogously, a state vector $\mathbf{x}$ with a negative element $s_{j_i}^{x_i} \leqslant x_i < 0$ has a membership $\prod_k \mu_{j_k}^{x_k}(x_k) > 0$ to core position vectors with the component $s_{j_i}^{x_i} < 0$ and a membership $\prod_k \mu_{j_k}^{x_k}(x_k) = 0$ to the core position vectors with the component $s_{j_i}^{x_i} > 0$. This yields in the sum over all $\mathbf{j} \in J^-(i)$ the inequality $0 < \sum_{\mathbf{j}\in J^-(i)} s_{w_i(\mathbf{j})}^{\dot{x}_i} \prod_k \mu_{jk}^{x_k}(x_k) < \infty$.

Using the results above, applying Eq. (11), and applying the triangle inequality results in the following estimation:

$$
\begin{aligned}
|x_i + \Delta t \dot{x}_i| &= \left| x_i + \Delta t \left[\sum_{\mathbf{j} \in J^-(i)} s^{\dot{x}_i}_{w_i(\mathbf{j})} \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) + \sum_{\mathbf{j} \in J^+(i)} s^{\dot{x}_i}_{w_i(\mathbf{j})} \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) \right] \right| \\
&\leqslant \left| x_i + \Delta t \sum_{\mathbf{j} \in J^-(i)} s^{\dot{x}_i}_{w_i(\mathbf{j})} \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) \right| + \Delta t \sum_{\mathbf{j} \in J^+(i)} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) \\
&= |x_i| - \Delta t \sum_{\mathbf{j} \in J^-(i)} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) + \Delta t \sum_{\mathbf{j} \in J^+(i)} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k),
\end{aligned}
\tag{12}
$$

where $0 \leqslant \Delta t < \varepsilon$ with a suitably small $\varepsilon > 0$ for all Δt.

We utilize Eq. (12) to estimate the Dini-derivative in Eq. (10) within the region U_0. This leads to

$$
\begin{aligned}
\bar{D}^+ v(\mathbf{x}(t)) &= \limsup_{\Delta t \to 0^+} \frac{\sum_{i \in I} [|x_i + \Delta t \dot{x}_i| - |x_i|]}{\Delta t} \\
&\leqslant \limsup_{\Delta t \to 0^+} \frac{\sum_{i \in I} [|x_i| + \Delta t \sum_{\mathbf{j} \in J^+(i)} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) - \Delta t \sum_{\mathbf{j} \in J^-(i)} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) - |x_i|]}{\Delta t} \\
&= \sum_{i \in I} \left[\sum_{\mathbf{j} \in J^+(i)} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) - \sum_{\mathbf{j} \in J^-(i)} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) \right] \\
&= \sum_{\mathbf{j} \in J} \left[\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) - \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) \right] \\
&= \sum_{\mathbf{j} \in J} \left[\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| - \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \right] \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k),
\end{aligned}
\tag{13}
$$

where $I^+(\mathbf{j}) = \{i | s^{x_i}_{j_i} \cdot s^{\dot{x}_i}_{w_i(\mathbf{j})} \geqslant 0\}$, $I^-(\mathbf{j}) = \{i | s^{x_i}_{j_i} \cdot s^{\dot{x}_i}_{w_i(\mathbf{j})} < 0\}$, and $I = \{1, \ldots, n\} = I^+(\mathbf{j}) \cup I^-(\mathbf{j})$. Note, the set I is independent of the index $\mathbf{j}$.

If the condition

$$
\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \leqslant \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}|
\tag{14}
$$

is satisfied for all $3^n - 1$ core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, associated with the core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, on the boundary, δU_e, of the elementary neighborhood of the equilibrium point, $\mathbf{x}_e$, then the estimation (13) yields $\bar{D}^+ v(\mathbf{x}(t)) \leqslant 0$ for all $\mathbf{x} \in U_0$. In this case the equilibrium point, $\mathbf{x}_e$, is stable in the sense of Lyapunov. □

By modification of Theorem 2, i.e. by introducing a little more strict condition for the core position gradients associated with the core positions that are adjacent to the equilibrium point, we obtain the following theorem on asymptotic stability.

Theorem 3 (*Asymptotic stability*). *Let the core position* $\mathbf{x}_e = \mathbf{s}^{\mathbf{x}}_{\mathbf{j_0}} = \mathbf{0}$ *be an equilibrium point of the normalized continuous-time recurrent fuzzy system*, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. *This equilibrium point is asymptotically stable if*

(1) *all* $2n$ *core position gradients*, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, *whose associated core position vectors*, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, *are adjacent to the equilibrium point*, *satisfy the condition*

$$
\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| < \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}|,
$$

(2) *and all* $3^n - 1 - 2n$ *core position gradients,* $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, *whose associated core position vectors,* $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, *are not adjacent to the equilibrium point but lie on the boundary of the elementary neighborhood of* $\mathbf{x}_e$, *satisfy the condition*

$$\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \leqslant \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}|,$$

where $I^+(\mathbf{j}) = \{i | s^{x_i}_{j_i} \cdot s^{\dot{x}_i}_{w_i(\mathbf{j})} \geqslant 0\}$ *and* $I^-(\mathbf{j}) = \{i | s^{x_i}_{j_i} \cdot s^{\dot{x}_i}_{w_i(\mathbf{j})} < 0\}$.

Proof. Theorem 3 will be proven similarly to the proof of Theorem 2. Note, here the condition $\bar{D}^+ v(\mathbf{x}(t)) < 0$, i.e. employing Eq. (13) the inequality

$$\bar{D}^+ v(\mathbf{x}(t)) \leqslant \sum_{\mathbf{j} \in J} \left[\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| - \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \right] \prod_{k=1}^{n} \mu^{x_k}_{j_k}(x_k) < 0 \tag{15}$$

must be satisfied. According to Theorem 3, all $3^n - 1 - 2n$ core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, that are not adjacent to the equilibrium point, $\mathbf{x}_e$, but lie on the boundary, δU_e, of the elementary neighborhood satisfy the condition

$$\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| \leqslant \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}|. \tag{16}$$

Hence, if $\prod_k \mu^{x_k}_{j_k}(x_k) > 0$ these core position vectors result in summands in Eq. (15) that are zero or negative.

Furthermore, for all $2n$ core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, that are adjacent to the equilibrium point, $\mathbf{x}_e$, and lie on the boundary, δU_e, of its elementary neighborhood we have according to Theorem 3 that

$$\sum_{i \in I^+(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}| < \sum_{i \in I^-(\mathbf{j})} |s^{\dot{x}_i}_{w_i(\mathbf{j})}|. \tag{17}$$

If $\prod_k \mu^{x_k}_{j_k}(x_k) > 0$, the summands in Eq. (15) that result from the core position vectors satisfying Eq. (17) take on negative values.

Because any state vector, $\mathbf{x}$, within the elementary neighborhood, U_e, has always a membership $\prod_k \mu^{x_k}_{j_k}(x_k) > 0$ to at least one core position vector, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, on the boundary, δU_e, that is adjacent to the equilibrium point and its core position gradient, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, satisfying inequality (17), there exists at least one negative summand in Eq. (15) for any state vector $\mathbf{x}$ within U_e.

The membership of a state vector $\mathbf{x}$ to the remaining core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$ on the boundary, δU_e, is $\prod_k \mu^{x_k}_{j_k}(x_k) \geqslant 0$ and the associated core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, also satisfy either inequality (16) or (17), such that they yield summands being zero or negative in Eq. (15) for any state vector $\mathbf{x}$ within U_e. Thus, the sum of all summands in Eq. (15) holds $\bar{D} v(\mathbf{x}(t)) < 0$ for all $\mathbf{x} \in U_0$ and the equilibrium point is asymptotically stable. □

The stability theorems are illustrated for two-dimensional systems in Fig. 6. The equilibrium point of system (a) is stable in the sense of Lyapunov. The gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, that are associated with the vertices of the region U_0 point to the inside of U_0 or along the boundary of U_0. The core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, of the remaining core position vectors point into the dotted region, along its boundary, or have the value zero, i.e. they are equilibrium points as well. Therefore, all core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, satisfy the condition of Theorem 2. The equilibrium point of system (b) is asymptotically stable. The core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w}(\mathbf{j})}$, associated with the vertices of the region U_0 only point inside U_0, satisfying Condition (1) of Theorem 3. Condition (2) of Theorem 3 holds for the other core position vectors.

3.2. Region of asymptotic stability

The proof of Theorem 3 shows, that Conditions (1)–(3) with $\bar{D}^+ v(\mathbf{x}(t)) < 0$ of Theorem 7 are satisfied within the region

$$U_0 = \{\mathbf{x} | v(\mathbf{x}) \leqslant 1\} = \left\{ \mathbf{x} \left| \sum_{i \in I} |x_i| \leqslant 1 \right. \right\},$$

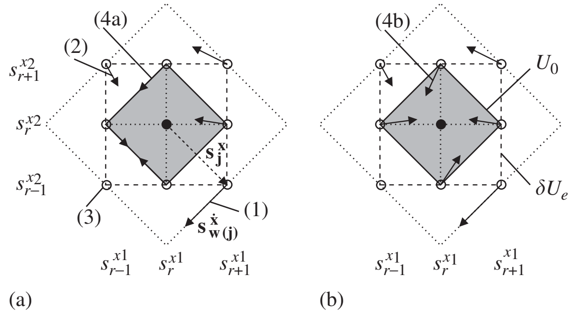

Fig. 6. The equilibrium point "•" of the two-dimensional system (a) is stable in the sense of Lyapunov and of system (b) is asymptotically stable. The boundary δU_e (dashed line) of U_e and the region U_0 (grey) are shown. The conditions of Theorems 2 or 3 are satisfied, e.g. (1) $|s^{\dot{x}2}_{w2(\mathbf{j})}| \leqslant |s^{\dot{x}1}_{w1(\mathbf{j})}|$, (2) $0 \leqslant |s^{\dot{x}1}_{w1(\mathbf{j})}| + |s^{\dot{x}2}_{w2(\mathbf{j})}|$, (3) $0 \leqslant |s^{\dot{x}1}_{w1(\mathbf{j})}| + |s^{\dot{x}2}_{w2(\mathbf{j})}| = 0$, (4a) $|s^{\dot{x}1}_{w1(\mathbf{j})}| \leqslant |s^{\dot{x}2}_{w2(\mathbf{j})}|$, (4b) $|s^{\dot{x}1}_{w1(\mathbf{j})}| < |s^{\dot{x}2}_{w2(\mathbf{j})}|$.

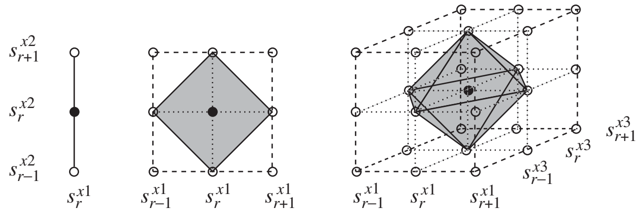

Fig. 7. Illustration of an equilibrium point "•" with the associated positively invariant set U_0 (grey) for one-, two-, and three-dimensional systems.

where $I = \{1, \ldots, n\}$. Thus, any state trajectory starting within the region U_0 stays within the region U_0 and ends at the equilibrium point, $\mathbf{x}_e$. Such a region is called a positively invariant set [27,20,7,35] or Lyapunov region. The largest possible set that includes all trajectories running into the equilibrium point $\mathbf{x}_e$ is defined as region of asymptotic stability [20,17].

The region of asymptotic stability of an equilibrium point whose asymptotic stability can be determined employing Theorem 3 has at least the size of the corresponding positively invariant set, U_0. Fig. 7 illustrates the minimum region of asymptotic stability of an equilibrium point that is asymptotically stable according to Theorem 3 for one-, two-, and three-dimensional systems, respectively. The vertices of U_0 correspond to the core position vectors that are adjacent to the equilibrium point.

Depending on the core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w(j)}}$, of the core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, lying outside the elementary neighborhood, U_e, of the equilibrium point, $\mathbf{x}_e$, the region of asymptotic stability may be larger than U_0.

3.3. *Existence of equilibrium points in elementary hypersquares*

In addition to the considerations above, it is desirable to know if equilibrium points other than those on core position vectors exist, i.e. within elementary hypersquares. For this purpose we can state the following existence theorems for this kind of equilibrium points.

Theorem 4. *Let* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *be a continuous-time recurrent fuzzy system, where the state vector,* $\mathbf{x} = \mathbf{0}$, *lies within an elementary hypersquare, H. If all core position vectors,* $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, *that are vertices of H, and their associated core position gradients,* $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w(j)}}$, *satisfy either* $s^{x_i}_{j_i} \cdot s^{\dot{x}_i}_{w_i(\mathbf{j})} \leqslant 0 \ \forall i \in \{1, \ldots, n\}$ *or* $s^{x_i}_{j_i} \cdot s^{\dot{x}_i}_{w_i(\mathbf{j})} \geqslant 0 \ \forall i \in \{1, \ldots, n\}$, *then at least one equilibrium point within H exists.*

Note, $s^{x_i}_{j_i} \neq 0$ holds for all i in Theorem 4.

Proof. We accomplish the proof by employing the fixed point theorem of Brouwer [40]. An elementary hypersquare is a non-empty, closed, convex set of a finite-dimensional normed space such that it satisfies the preconditions of Brouwer's Theorem.

We start off by rewriting the component, f_i, of the function, $\mathbf{f}$, with respect to the component, x_i, of the state vector, $\mathbf{x}$, according to Eq. (A.5) of the Appendix A.2, i.e.

$$\dot{x}_i = f_i(\mathbf{x}) = [f_i(\mathbf{x}_{x_i=s_r^{x_i}}) - f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}})]\mu_r^{x_i}(x_i) + f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}}), \tag{18}$$

where $s_r^{x_i}$ and $s_{r+1}^{x_i}$ are two adjacent core positions, the component, x_i, satisfies $s_r^{x_i} \leqslant x_i \leqslant s_{r+1}^{x_i}$, $\mathbf{x}_{x_i=s_r^{x_i}} = [x_1, \ldots, x_{i-1}, s_r^{x_i}, x_{i+1}, \ldots, x_n]^{\mathrm{T}}$, and $\mathbf{x}_{x_i=s_{r+1}^{x_i}} = [x_1, \ldots, x_{i-1}, s_{r+1}^{x_i}, x_{i+1}, \ldots, x_n]^{\mathrm{T}}$. The equilibrium points are given by

$$\dot{x}_i = [f_i(\mathbf{x}_{x_i=s_r^{x_i}}) - f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}})]\mu_r^{x_i}(x_i) + f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}}) = 0 \quad \forall i \in \{1, \ldots, n\}. \tag{19}$$

We distinguish two cases (a) $f_i(\mathbf{x}_{x_i=s_r^{x_i}}) = f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}}) = 0$ and (b) $f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}}) \neq 0$ and/or $f_i(\mathbf{x}_{x_i=s_r^{x_i}}) \neq 0$. Each component i belongs to one of these two cases. By employing the triangular membership function $\mu_r^{x_i}(x_i) = 1 - (x_i - s_r^{x_i})/(s_{r+1}^{x_i} - s_r^{x_i})$ within a hypersquare, H, of a standardized continuous-time recurrent fuzzy system, we can rewrite Eq. (19) according to the cases (a) and (b) as follows:

$$\text{(a)} \quad x_i = x_i = h_i(\mathbf{x}), \tag{20}$$

$$\text{(b)} \quad x_i = \frac{f_i(\mathbf{x}_{x_i=s_r^{x_i}})}{f_i(\mathbf{x}_{x_i=s_r^{x_i}}) - f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}})}(s_{r+1}^{x_i} - s_r^{x_i}) + s_r^{x_i} = h_i(\mathbf{x}). \tag{21}$$

Combining the Eqs. (20), (21) of all n components, x_i, of a state vector, $\mathbf{x}$, in one vector leads to a system of equations, $\mathbf{x} = \mathbf{h}(\mathbf{x})$. Every solution of $\mathbf{x} = \mathbf{h}(\mathbf{x})$ also fulfills the corresponding fixed point equation $\mathbf{x}_{k+1} = \mathbf{h}(\mathbf{x}_k)$ with $\mathbf{x}_{k+1} = \mathbf{x}_k$. Thus, all equilibrium points of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ are identical with the fixed points of $\mathbf{x}_{k+1} = \mathbf{h}(\mathbf{x}_k)$.

In the following, we verify that the function, $\mathbf{h}$, fulfills the conditions of the fixed point theorem of Brouwer, i.e. it maps the elementary hypersquare into itself.

The condition $s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} \leqslant 0$ for all core position vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, leads to the inequalities $s_r^{x_i} \cdot f_i(\mathbf{x}_{x_i=s_r^{x_i}}) \leqslant 0$ and $s_{r+1}^{x_i} \cdot f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}}) \leqslant 0$. Since the state vector $\mathbf{x} = \mathbf{0}$ lies within the elementary hypersquare, i.e. $s_r^{x_i} < 0$ and $s_{r+1}^{x_i} > 0$, we have that $f_i(\mathbf{x}_{x_i=s_r^{x_i}}) \geqslant 0$ and $f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}}) \leqslant 0$,which yields $0 \leqslant f_i(\mathbf{x}_{x_i=s_r^{x_i}})/(f_i(\mathbf{x}_{x_i=s_r^{x_i}}) - f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}})) \leqslant 1$. Analogously, if all core position vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, satisfy the condition $s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} \geqslant 0$, we get $f_i(\mathbf{x}_{x_i=s_r^{x_i}}) \leqslant 0$ and $f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}}) \geqslant 0$, which again yields $0 \leqslant f_i(\mathbf{x}_{x_i=s_r^{x_i}})/(f_i(\mathbf{x}_{x_i=s_r^{x_i}}) - f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}})) \leqslant 1$.

Thus, the components h_i of the function $\mathbf{h}$ are bounded by

$$\text{(a)} \quad s_r^{x_i} \leqslant h_i(\mathbf{x}) = x_i \leqslant s_{r+1}^{x_i}, \tag{22}$$

$$\text{(b)} \quad s_r^{x_i} \leqslant h_i(\mathbf{x}) = \frac{f_i(\mathbf{x}_{x_i=s_r^{x_i}})}{f_i(\mathbf{x}_{x_i=s_r^{x_i}}) - f_i(\mathbf{x}_{x_i=s_{r+1}^{x_i}})}(s_{r+1}^{x_i} - s_r^{x_i}) + s_r^{x_i} \leqslant s_{r+1}^{x_i}. \tag{23}$$

Obviously, the function, $\mathbf{h}$, maps any state vector, $\mathbf{x}$, from the interior of the elementary hypersquare into the elementary hypersquare itself and consequently, satisfies the conditions of the fixed point theorem of Brouwer, i.e. the function, $\mathbf{h}$, has at least one fixed point within this elementary hypersquare. As a result, at least one equilibrium point of the system, $\mathbf{f}$, exists within this elementary hypersquare. □

If the state vector $\mathbf{x} = \mathbf{0}$ does not lie within the considered elementary hypersquare, it is possible to transform the system appropriately without loss of generality.

Condition $s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} \leqslant 0$ of Theorem 4 provides that the gradients $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}}$ of all vertices of the elementary hypersquare point to its inside or along its boundary. Fig. 8 depicts two elementary hypersquares with at least one equilibrium point according to Theorem 4.

Besides the above considered case, where at least one equilibrium point exists within an elementary hypersquare, it is possible that all state vectors, $\mathbf{x}$, within an elementary hypersquare, H, are equilibrium points of a continuous-time recurrent fuzzy system, $\mathbf{f}$. The following Lemma gives a formal description:

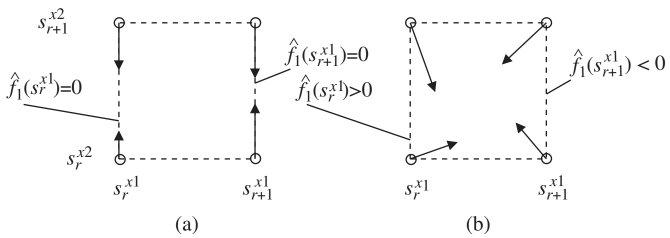

Fig. 8. Schematic of an elementary hypersquare of a two-dimensional system. E.g. considering coordinate x_1 yields (a): $\hat{f}_1(s_r^{x1}) = \hat{f}_1(s_{r+1}^{x1}) = 0$ and (b): $\hat{f}_1(s_r^{x1}) \geqslant 0$ and $\hat{f}_1(s_{r+1}^{x1}) \leqslant 0$. The conditions of Theorem 4 are satisfied, such that there exists at least one equilibrium point for each of the systems (a) and (b).

Lemma 1. *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuous-time recurrent fuzzy system with an elementary hypersquare, H. If all core position vectors, $\mathbf{s}_\mathbf{j}^\mathbf{x}$, being vertices of H are equilibrium points of $\mathbf{f}$, then all state vectors, $\mathbf{x} \in H$, are equilibrium points of H.*

Proof. Since all core position vectors, $\mathbf{s}_\mathbf{j}^\mathbf{x}$, that are vertices of H, are equilibrium points of $\mathbf{f}$, they have associated core position gradients $\mathbf{s}_{\mathbf{w(j)}}^{\dot{\mathbf{x}}} = \mathbf{0}$. All state vectors, $\mathbf{x} \in H$, have gradients, $\dot{\mathbf{x}}$, solely interpolated from the core position gradients, $\mathbf{s}_{\mathbf{w(j)}}^{\dot{\mathbf{x}}}$, associated with the vertices, $\mathbf{s}_\mathbf{j}^\mathbf{x}$, of H, i.e. they have a membership $\prod_i \mu_{j_i}^{x_i}(x_i) > 0$ to the vertices, $\mathbf{s}_\mathbf{j}^\mathbf{x}$, of H and a membership $\prod_i \mu_{j_i}^{x_i}(x_i) = 0$ to any other core position vector. The set $J_H(\mathbf{x}) = \{\mathbf{j} | \prod_i \mu_{j_i}^{x_i}(x_i) > 0\}$ includes the index vectors of the core position gradients, $\mathbf{s}_\mathbf{j}^\mathbf{x}$, being vertices of H. For all state vectors $\mathbf{x} \in H$ this leads to

$$\dot{\mathbf{x}} = \sum_{\mathbf{j}} \mathbf{s}_{\mathbf{w(j)}}^{\dot{\mathbf{x}}} \prod_{i=1}^{n} \mu_{j_i}^{x_i}(x_i) = \sum_{\mathbf{j} \in J_H} \mathbf{0} \cdot \prod_{i=1}^{n} \mu_{j_i}^{x_i}(x_i) + \sum_{\mathbf{j} \notin J_H} \mathbf{s}_{\mathbf{w(j)}}^{\dot{\mathbf{x}}} \cdot 0 = \mathbf{0}. \qquad \square$$

We have so far considered the existence of equilibrium points within elementary hypersquares. The following sufficient theorem yields a statement to exclude equilibrium points from elementary hypersquares.

Theorem 5. *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuous-time recurrent fuzzy system. If a common polyhedral cone,*

$$P(\mathbf{S}_n) = \{\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n,\ \mathbf{S}_n\mathbf{x} > \mathbf{0}\} \neq \emptyset \quad \textit{with } \mathbf{S}_n = [\mathbf{s}_{\mathbf{w}(\mathbf{j}_1)}^{\dot{\mathbf{x}}}, \ldots, \mathbf{s}_{\mathbf{w}(\mathbf{j}_m)}^{\dot{\mathbf{x}}}]^\mathrm{T} \in \mathbb{R}^{m \times n},$$

exists for the core position gradients, $\mathbf{s}_{\mathbf{w}(\mathbf{j}_l)}^{\dot{\mathbf{x}}}$, of all m core position vectors, $\mathbf{s}_{\mathbf{j}_l}^\mathbf{x}$ with $\mathbf{j}_l \in \{1, \ldots, m\}$, that are vertices of an elementary hypersquare, H, of the system, $\mathbf{f}$, then no equilibrium points exist in H.

Proof. If a polyhedral cone,

$$P(\mathbf{S}_n) = \{\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n,\ \mathbf{S}_n\mathbf{x} > \mathbf{0}\} \neq \emptyset \quad \text{with } \mathbf{S}_n = [\mathbf{s}_{\mathbf{w}(\mathbf{j}_1)}^{\dot{\mathbf{x}}}, \ldots, \mathbf{s}_{\mathbf{w}(\mathbf{j}_m)}^{\dot{\mathbf{x}}}]^\mathrm{T} \in \mathbb{R}^{m \times n},$$

exists according to Theorem 5, then the elementary hypersquare can be represented in a rotated coordinate system with the axes $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_n$, where one axis $\tilde{\mathbf{x}}_i$ satisfies $\tilde{\mathbf{x}}_i \in P(\mathbf{S}_n)$.

In the following, we define $\mathbf{j} \in \{\mathbf{j}_1, \ldots, \mathbf{j}_m\}$. Since all m core position gradients, $\mathbf{s}_{\mathbf{w(j)}}^{\dot{\mathbf{x}}}$, associated with the vertices, $\mathbf{s}_\mathbf{j}^\mathbf{x}$, of the elementary hypersquare H yield $(\mathbf{s}_{\mathbf{w(j)}}^{\dot{\mathbf{x}}})^\mathrm{T} \cdot \tilde{\mathbf{x}}_i > 0$, the gradients $\tilde{\mathbf{s}}_{\mathbf{w(j)}}^{\dot{\tilde{\mathbf{x}}}}$ in the rotated coordinate system satisfy the inequality $\tilde{s}_{w_i(\mathbf{j})}^{\dot{\tilde{x}}_i} > 0$ in the i-th component. Together with a membership $\prod_k \tilde{\mu}_{j_k}^{\tilde{x}_k}(\tilde{x}_k) > 0$ within the hypersquare, H, the component $\dot{\tilde{x}}_i$ of the recurrent fuzzy system in the rotated coordinate system can be estimated to

$$\dot{\tilde{x}}_i = \tilde{f}_i(\tilde{\mathbf{x}}) = \sum_{\mathbf{j}} \tilde{s}_{w_i(\mathbf{j})}^{\dot{\tilde{x}}_i} \prod_{k=1}^{n} \tilde{\mu}_{j_k}^{\tilde{x}_k}(\tilde{x}_k) > 0.$$

Therefore, equilibrium points can be excluded from H. Since rotating the coordinate system leaves the dynamics of the system unchanged, equilibrium points in the original system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, can also be excluded. $\square$

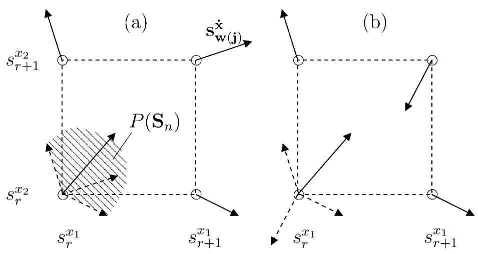

Fig. 9. Two elementary hypersquares are presented with the core position gradients $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w(j)}}$. (a) A common polyhedral cone, $P(\mathbf{S}) \neq \emptyset$, exists for the core position gradients $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w(j)}}$. (b) No common polyhedral cone, $P(\mathbf{S})$, exists for the core position gradients, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w(j)}}$.

Whether such a polyhedral cone, $P(\mathbf{S}_n) \neq \emptyset$, exists can be determined using the following necessary and sufficient theorem that comprises the well-known Fourier–Motzkin elimination [46], $\mathbf{F}_{n-1}(\mathbf{S}_n)$, of a matrix, $\mathbf{S}_n$, with $\dim \mathbf{S}_n = m \times n$ and $\dim \mathbf{F}_{n-1}(\mathbf{S}_n) = \tilde{m} \times (n-1)$ specified in Appendix A.3.

Theorem 6 (*Xia [46]*). *$P(\mathbf{S}_n) = \emptyset$ iff the n-th column of the $m \times n$ matrix, $\mathbf{S}_n$, does not comprise solely either positive or negative elements and $P(\mathbf{F}_{n-1}(\mathbf{S}_n)) = \emptyset$.*

To determine whether a polyhedral cone, $P(\mathbf{S}_n) \neq \emptyset$, exists, Theorem 6 must be applied recursively, i.e. the Fourier–Motzkin elimination is applied repeatedly on the matrix $\mathbf{S}_n$ until in step h, with $1 \leqslant h \leqslant n-1$, the $(n-h)$-th column of the $\tilde{m} \times (n-h)$ matrix $\mathbf{S}_{n-h} = \mathbf{F}_{n-h}(\ldots \mathbf{F}_{n-2}(\mathbf{F}_{n-1}(\mathbf{S}_n)))$ comprises solely either positive or negative elements. Then, we have that $P(\mathbf{S}_{n-h}) \neq \emptyset$, which implies also $P(\mathbf{S}_{n-h+1}) \neq \emptyset$, $P(\mathbf{S}_{n-h+2}) \neq \emptyset, \ldots, P(\mathbf{S}_{n-1}) \neq \emptyset$, and $P(\mathbf{S}_n) \neq \emptyset$, i.e. a polyhedral cone exists.

If also step $h = n-1$ does not lead to an $\tilde{m} \times 1$ matrix $\mathbf{S}_1$ with solely either positive or negative elements, $P(\mathbf{S}_1) = \emptyset$ can be concluded. Since all other matrices $\mathbf{S}_2, \ldots, \mathbf{S}_n$ have also both positive and negative elements in their last column we recursively conclude that $P(\mathbf{S}_2) = \emptyset$, $P(\mathbf{S}_3) = \emptyset, \ldots, P(\mathbf{S}_{n-1}) = \emptyset$, and $P(\mathbf{S}_n) = \emptyset$ according to Theorem 6. Thus, a polyhedral cone does not exist [46].

Fig. 9 depicts two systems, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. A polyhedral cone exists for system (a) whereas a polyhedral cone does not exist for system (b).

4. Qualitative and quantitative model

Summarizing the preceding sections, modeling dynamics using continuous-time recurrent fuzzy systems is suitable for systems whose dynamics are linguistically describable in nodes. According to Section 2, modeling is organized in two steps: (1) qualitative modeling and (2) quantitative modeling. The quantitative model facilitates a detailed mathematical analysis of the system dynamics and their simulation.

(1) The first step, the qualitative modeling, leads to a linguistic description of the system dynamics. This requires at least fuzzy knowledge of the dynamic behavior in all nodes, or more precise: one must be able to describe linguistically the state vector, $\mathbf{x}$, and the input vector, $\mathbf{u}$, by the linguistic vectors, $\mathbf{L}^{\mathbf{x}}_{\mathbf{j}}$ and $\mathbf{L}^{\mathbf{u}}_{\mathbf{q}}$, and the variation, $\dot{\mathbf{x}}$, of the state vector, $\mathbf{x}$, by the linguistic gradient, $\mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{w(j,q)}}$, in these nodes. This leads to a qualitative model describing the dynamics of the system in nodes using if–then rules of the following structure:

$$\text{If } \mathbf{x}(t) = \mathbf{L}^{\mathbf{x}}_{\mathbf{j}} \quad \text{and} \quad \mathbf{u}(t) = \mathbf{L}^{\mathbf{u}}_{\mathbf{q}}, \qquad \text{then } \dot{\mathbf{x}}(t) = \mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{w(j,q)}}.$$

(2) In the second step the qualitative model is employed to gain a quantitative model. For this purpose measured or representative values have to be assigned to the linguistic description of the dynamics in the nodes from step (1). Thus, the linguistic vectors, $\mathbf{L}^{\mathbf{x}}_{\mathbf{j}}$ and $\mathbf{L}^{\mathbf{u}}_{\mathbf{q}}$, of each node are represented by numerical values defined as the core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$ and $\mathbf{s}^{\mathbf{u}}_{\mathbf{q}}$, in the state space. Similarly, a core position gradient, $\mathbf{s}^{\dot{\mathbf{x}}}_{\mathbf{w(j,q)}}$, associated with each node, $(\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}, \mathbf{s}^{\mathbf{u}}_{\mathbf{q}})$, numerically represents the linguistic gradient, $\mathbf{L}^{\dot{\mathbf{x}}}_{\mathbf{w(j,q)}}$, in the corresponding node. This numeric gradient is obtained through measurement of the variation of the states in the nodes or through representative choice. The derivative, $\dot{\mathbf{x}}$, of

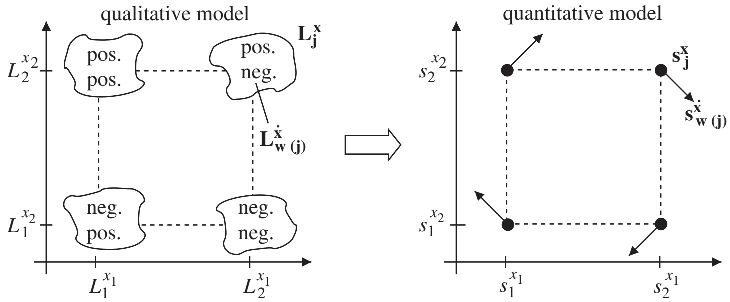

Fig. 10. The qualitative model provides a linguistic description of the dynamics of the system. The quantitative fuzzy model employs numerical values, i.e. the core positions and core position gradients, and interpolates the dynamics in between the core positions.

states, $\mathbf{x}$, between nodes $(\mathbf{s}_\mathbf{j}^\mathbf{x}, \mathbf{s}_\mathbf{q}^\mathbf{u})$ is interpolated by means of the fuzzy system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \sum_{\mathbf{j},\mathbf{q}} \mathbf{s}_{\mathbf{w}(\mathbf{j},\mathbf{q})}^{\dot{\mathbf{x}}} \prod_{i=1}^{n} \mu_{j_i}^{x_i}(x_i) \prod_{p=1}^{m} \mu_{q_p}^{u_p}(u_p)$$

with $\mu_{j_i}^{x_i}(x_i)$ and $\mu_{q_p}^{u_p}(u_p)$ as membership functions of the state vector, $\mathbf{x}$, and input vector, $\mathbf{u}$, respectively.

Fig. 10 illustrates step (1) and (2). Modeling and investigation of the dynamics shall be clarified by demonstrating the procedures on a simple example in the next section.

5. Example: the fuzzy-Solow model

The following example is based on the extended Solow model that is well known in the field of economic theory and illustrates the general properties and possibilities of continuous-time recurrent fuzzy systems. The model supplies a simplified description of the dynamic relationship between human capital, H, (e.g. knowledge, experience and skills of specialists) and physical capital, K, (e.g. infrastructure) of an economic system with competitive markets and identity of all companies. Both the number of employees, L, and the level of technology, A, of the economy grow with a constant rate. The product $A \cdot L$ defines a measure for an effective worker to which all variables of the model are normalized, making it independent of A and L [9,21]. In the following, these normalized variables are denoted with the corresponding small letters.

5.1. Qualitative modeling

The variation with respect to time of the human capital, h, per effective worker and the physical capital, k, per effective worker result from both a relative depreciation of h and k over time and investment in h and k depending on the share of income, y, per effective worker in the economy. In this model solely h and k determine the income, y, [9,21]. This leads to the following exemplary if–then rule:

If h is big and k is big, then the change in h is somewhat negative and the change in k is somewhat negative.

The complete rule base of the simplified qualitative model is shown in Table 2. By introducing additional rules the system can be modeled in a considerably more complex way.

5.2. Quantitative modeling

We obtain a quantitative model from the qualitative model description by assigning the core positions, $s_{j_i}^{x_i}$, and core position derivatives, $s_{w_i(\mathbf{j})}^{\dot{x}_i}$, shown in Table 3 to the linguistic values, $L_{j_i}^{x_i}$ and $L_{w_i(\mathbf{j})}^{\dot{x}_i}$, respectively from Table 2. To simplify matters, we assign normalized values in our example. The core positions, $s_{j_i}^{x_i}$, define triangular fuzzy membership functions and the core position derivatives, $s_{w_i(\mathbf{j})}^{\dot{x}_i}$, define singletons. The core position $s_{j_i}^{x_i} = -1$ e.g. represents the linguistic value $L_{j_i}^{x_i} = small$. The fuzzy system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, interpolates between the nodes.

Table 2
Rule matrices that represent the change in human capital $L_{j_1}^{x_1}$ and physical capital $L_{j_2}^{x_2}$ respectively

		Human capital $L_{j_1}^{x_1}$			Human capital $L_{j_1}^{x_1}$		
		Small	Medium	Big	Small	Medium	Big
Physical Capital $L_{j_2}^{x_2}$	Big	$+++$	$+$	$-$	$---$	$--$	$-$
	Medium	$++$	$\circ$	$--$	$-$	$\circ$	$+$
	Small	$+$	$-$	$---$	$+$	$++$	$+++$
		(a) Change in human capital $L_{w_1(\mathbf{j})}^{\dot{x}_1}$			(b) Change in physical capital $L_{w_2(\mathbf{j})}^{\dot{x}_2}$		

$+++$ = strongly positive, $++$ = positive, $+$ = somewhat positive, $\circ$ = zero, $-$ = somewhat negative, $--$ = negative, $---$ = strongly negative.

Table 3
Assignment of core positions $s_{j_i}^{x_i}$ and core position derivatives $s_{w_i(\mathbf{j})}^{\dot{x}_i}$ to linguistic values $L_{j_i}^{x_i}$ and linguistic derivatives $L_{w_i(\mathbf{j})}^{\dot{x}_i}$, respectively

$L_{j_i}^{x_i}$	Small	Medium	Big				
$s_{j_i}^{x_i}$	-1	0	1				
$L_{w_i(\mathbf{j})}^{\dot{x}_i}$	$+++$	$++$	$+$	$\circ$	$-$	$--$	$---$
$s_{w_i(\mathbf{j})}^{\dot{x}_i}$	1.5	1	0.5	0	-0.5	-1	-1.5

5.3. Investigation of the system dynamics

Employing the quantitative model description and Theorem 1 shows that the core position vector $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} = [0,\ 0]^{\mathrm{T}}$ with the associated gradient $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}} = [0,\ 0]^{\mathrm{T}}$ is an equilibrium point of the system.

The stability of the equilibrium point is investigated by applying Theorem 3. The four core position vectors, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$, that are adjacent to the equilibrium point have to satisfy Condition (1) of Theorem 3. As an example, we consider the core position vector $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} = [0,\ 1]^{\mathrm{T}}$ and its associated core position gradient $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}} = [0.5, -1]^{\mathrm{T}}$. According to Theorem 3 the index sets for this core position vector are $I^+(\mathbf{j}) = \{i | s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} \geqslant 0\} = \{1\}$ and $I^-(\mathbf{j}) = \{i | s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} < 0\} = \{2\}$. Thus, Condition (1),

$$\sum_{i \in I^+(\mathbf{j})} |s_{w_i(\mathbf{j})}^{\dot{x}_i}| = |0.5| < \sum_{i \in I^-(\mathbf{j})} |s_{w_i(\mathbf{j})}^{\dot{x}_i}| = |-1|, \tag{24}$$

is satisfied.

To verify Condition (2) of Theorem 3 all core position vectors, that lie on the boundary, δU_e, of the elementary neighborhood associated with the equilibrium point and that are not adjacent to the equilibrium point have to be considered. In this case we consider the core position vector $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} = [-1,\ 1]^{\mathrm{T}}$ with the associated core position gradient $\mathbf{s}_{\mathbf{w}(\mathbf{j})}^{\dot{\mathbf{x}}} = [1.5,\ -1.5]^{\mathrm{T}}$ as an example. According to Theorem 3 this leads to the index sets $I^+(\mathbf{j}) = \{i | s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} \geqslant 0\} = \emptyset$ and $I^-(\mathbf{j}) = \{i | s_{j_i}^{x_i} \cdot s_{w_i(\mathbf{j})}^{\dot{x}_i} < 0\} = \{1,\ 2\}$ so that the inequality

$$\sum_{i \in I^+(\mathbf{j})} |s_{w_i(\mathbf{j})}^{\dot{x}_i}| = 0 \leqslant \sum_{i \in I^-(\mathbf{j})} |s_{w_i(\mathbf{j})}^{\dot{x}_i}| = |1.5| + |-1.5| = 3 \tag{25}$$

is satisfied. Similar to Eqs. (24), (25) we can prove for all the remaining core position vectors that the Conditions (1) and (2) of Theorem 3 are satisfied. Thus, the equilibrium point $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} = [0,\ 0]^{\mathrm{T}}$ is asymptotically stable.

According to Section 3.2, the region of asymptotic stability of the equilibrium point corresponds at least to the positively invariant set whose vertices match with the core position vectors that are adjacent to the equilibrium point. Fig. 11(a) illustrates the system dynamics in the state space. Fig. 11(b) depicts a plot of h and k for the case of an initial state $\mathbf{x} = [0,\ 1]^{\mathrm{T}}$.

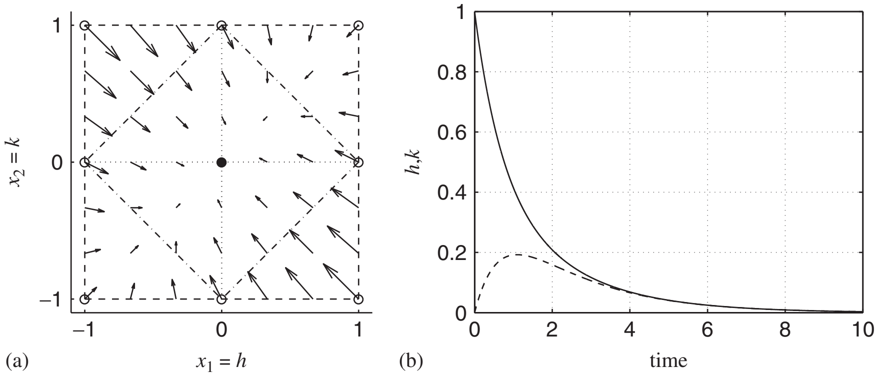

Fig. 11. (a) Illustration in the state space of the dynamics of the example with an asymptotically stable equilibrium point (•), region of asymptotic stability (— · —), and elementary neighborhood (- -). Arrows represent the gradients $\dot{\mathbf{x}}$ of some states $\mathbf{x}$. (b) Plot of h (- -) and k (–) over time for the initial value $\mathbf{x} = [0,\ 1]^T$.

6. Summary and conclusions

In this article we linguistically and mathematically describe continuous-time recurrent fuzzy systems on the basis of discrete-time recurrent fuzzy systems [1,3,25,26] and investigate their dynamics. They are suited to describe continuous-time processes whose states and variation of the states are hard to be measured quantitatively, but can easily be described qualitatively employing linguistic if–then rules. Systems of the same structure were already introduced in [5] and utilized for interpolation between nodes. Their dynamic properties have not been investigated systematically so that this article covering the investigation of equilibrium points provides a first step towards the theory of these systems.

In addition to equilibrium points, continuous-time recurrent fuzzy systems may have further dynamic properties which include e.g. oscillations and limit cycles. The investigation of such properties remains a challenge for further research.

Appendix A

A.1. Lyapunov's stability theorem for non-differentiable Lyapunov functions

Lyapunov's stability theorem for non-differentiable Lyapunov functions [36] is employed in the proofs of the stability Theorems 2 and 3.

Theorem 7. *The differential equation*, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *with a continuous function* $\mathbf{f}$, *having an equilibrium point*, $\mathbf{x} = \mathbf{0}$, *has a unique solution for every initial state taken from a neighborhood*, U_1, *of the origin. The function* $v(\mathbf{x})$ *is lipschitzian with respect to* $\mathbf{x}$ *and continuous in a neighborhood* $U_0 \subseteq U_1$. *If* $v(\mathbf{x})$ *satisfies the condition*

(1) $v(\mathbf{0}) = 0$, *and with the exception of* $\mathbf{x} = \mathbf{0}$ *the conditions*
(2) $v(\mathbf{x}) > 0$,
(3) $\bar{D}^+ v(\mathbf{x}(t)) = \limsup_{\Delta t \to 0+} \dfrac{v(\mathbf{x}(t) + \Delta t \dot{x}(t)) - v(\mathbf{x}(t))}{\Delta t} < 0$ *and* $\bar{D}^+ v(\mathbf{0}) = 0$,

then the equilibrium point $\mathbf{x} = \mathbf{0}$ *is asymptotically stable and* $v(\mathbf{x})$ *will be called a "Lyapunov function." If Condition* (3) *is* $\bar{D}^+ v(\mathbf{x}(t)) \leqslant 0$, *then the equilibrium point* $\mathbf{x} = \mathbf{0}$ *is stable in the sense of Lyapunov.*

A.2. Alternative representation of recurrent fuzzy systems

According to [24] the components, $f_i(\mathbf{x})$, of the function, $\mathbf{f}(\mathbf{x})$, appearing in Eq. (6) can be represented with respect to the component, x_h, of the state vector, $\mathbf{x}$, within an elementary hypersquare, H. Therefore, we rewrite the components

$f_i(\mathbf{x})$ of Eq. (6) for the autonomous case to

$$f_i(\mathbf{x}) = \sum_{\hat{\mathbf{j}}} s^{\dot{x}_i}_{w_i(\hat{\mathbf{j}})} \prod_{k \neq h} \mu^{x_k}_{\hat{j}_k}(x_k) \cdot \mu^{x_h}_{\hat{j}_h}(x_h), \tag{A.1}$$

where the core positions are denoted by the index vector $\hat{\mathbf{j}}$ in this case. The term $\mu^{x_h}_{\hat{j}_h}(x_h)$ in the equation above is factored out explicitly from the product. If x_h lies on an arbitrary core position, $s^{x_h}_{j_h}$, then the membership to a core position with the index vector $\hat{\mathbf{j}}$ is given by $\mu^{x_h}_{\hat{j}_h}(s^{x_h}_{j_h}) = \delta_{\hat{j}_h, j_h}$, where $\delta_{\hat{j}_h, j_h}$ is the Kronecker delta. If $s^{x_h}_{\hat{j}_h}$ is identical to $s^{x_h}_{\hat{j}_h}$ we have $\hat{j}_h = j_h$ and therefore $\delta_{\hat{j}_h, j_h} = 1$ and if $s^{x_h}_{j_h}$ does not lie on the certain core position $s^{x_h}_{\hat{j}_h}$ we have $\hat{j}_h \neq j_h$ and therefore $\delta_{\hat{j}_h, j_h} = 0$. Thus for $x_h = s^{x_h}_{j_h}$, we obtain

$$f_i(\mathbf{x}_{x_h = s^{x_h}_{j_h}}) = \sum_{\hat{\mathbf{j}}} s^{\dot{x}_i}_{w_i(\hat{\mathbf{j}})} \prod_{k \neq h} \mu^{x_k}_{\hat{j}_k}(x_k) \cdot \delta_{\hat{j}_h, j_h}, \tag{A.2}$$

where $\mathbf{x}_{x_h = s^{x_h}_{j_h}} = [x_1, \dots, x_{h-1}, s^{x_h}_{j_h}, x_{h+1}, \dots, x_n]$. In the following we generate a sum of the terms $f_i(\mathbf{x}_{x_h = s^{x_h}_{j_h}})$ weighted with $\mu^{x_h}_{j_h}(x_h)$:

$$\sum_{j_h} f_i(\mathbf{x}_{x_h = s^{x_h}_{j_h}}) \cdot \mu^{x_h}_{j_h}(x_h) = \sum_{j_h} \left[\sum_{\hat{\mathbf{j}}} s^{\dot{x}_i}_{w_i(\hat{\mathbf{j}})} \prod_{k \neq h} \mu^{x_k}_{\hat{j}_k}(x_k) \cdot \delta_{\hat{j}_h, j_h} \right] \cdot \mu^{x_h}_{j_h}(x_h). \tag{A.3}$$

Due to the Kronecker delta, the summands with $j_h \neq \hat{j}_h$ of the sum $\sum_{j_h}$ on the right-hand side of Eq. (A.3) yield zero. The summands with $j_h = \hat{j}_h$ are not equal to zero. The sum of these non-zero summands is equal to the right-hand side of Eq. (A.1), such that we obtain

$$\sum_{j_h} \left[\sum_{\hat{\mathbf{j}}} s^{\dot{x}_i}_{w_l(\hat{\mathbf{j}})} \prod_{k \neq h} \mu^{x_k}_{\hat{j}_k}(x_k) \cdot \delta_{\hat{j}_h, j_h} \right] \cdot \mu^{x_h}_{j_h}(x_h) = \sum_{\hat{\mathbf{j}}} s^{\dot{x}_i}_{w_i(\hat{\mathbf{j}})} \prod_{k \neq h} \mu^{x_k}_{\hat{j}_k}(x_k) \cdot \mu^{x_h}_{\hat{j}_h}(x_h).$$

This leads to

$$f_i(\mathbf{x}) = \sum_{j_h} f_i(\mathbf{x}_{x_h = s^{x_h}_{j_h}}) \cdot \mu^{x_h}_{j_h}(x_h). \tag{A.4}$$

Within the considered elementary hypersquare, H, we have that $s^{x_h}_r \leqslant x_h \leqslant s^{x_h}_{r+1}$, where $s^{x_h}_r$ and $s^{x_h}_{r+1}$ are two adjacent core positions such that the membership function, $\mu^{x_h}_{j_h}(x_h)$, is positive for no more than the two indices $j_h = r$ and $j_h = r+1$. Furthermore, due to the conditions given in Section 2.2, we have $\mu^{x_h}_{r+1}(x_h) = 1 - \mu^{x_h}_r(x_h)$ within the elementary hypersquare, H. By employing this in Eq. (A.4) and computing the sum over $j_h \in \{r, r+1\}$, we obtain for Eq. (A.1)

$$\begin{aligned} f_i(\mathbf{x}) &= f_i(\mathbf{x}_{x_h = s^{x_h}_r})\mu^{x_h}_r(x_h) + f_i(\mathbf{x}_{x_h = s^{x_h}_{r+1}})\mu^{x_h}_{r+1}(x_h) \\ &= [f_i(\mathbf{x}_{x_h = s^{x_h}_r}) - f_i(\mathbf{x}_{x_h = s^{x_h}_{r+1}})]\mu^{x_h}_r(x_h) + f_i(\mathbf{x}_{x_h = s^{x_h}_{r+1}}). \end{aligned} \tag{A.5}$$

A.3. Fourier–Motzkin elimination

The following Fourier–Motzkin elimination, given e.g. in [46], is applied in Section 3.3 to exclude equilibrium points within an elementary hypersquare.

The matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ comprises the elements, a_{ij}. Associated with the matrix $\mathbf{A}$ we define the sets

$$Q = \{k | a_{kn} < 0\}, \quad P = \{k | a_{kn} > 0\} \quad \text{and} \quad Z = \{k | a_{kn} = 0\},$$

and the number $\tilde{m} = |Z| + |Q \times P|$. Here, the number $|M|$ denotes the number of elements in a finite set, M.

The matrix $\mathbf{B} = \mathbf{F}_{n-1}(\mathbf{A}) \in \mathbb{R}^{\tilde{m}\times(n-1)}$ with the elements b_{ij} is said to be the Fourier–Motzkin elimination of the matrix $\mathbf{A}$ and its elements are assigned as follows:

- For $i = 1, \ldots, |Z|$ we have that

$$b_{ij} = a_{i'j},$$

  where $j = 1, \ldots, n-1$ and $i' \in Z$. Thereby, only one index i corresponds to only one arbitrarily chosen index i'.
- For $i = |Z| + 1, \ldots, |P \times Q| + |Z|$ we have that

$$b_{ij} = a_{i'j} - \frac{a_{i'n}}{a_{k'n}} a_{k'j},$$

  where $j = 1, \ldots, n-1$ and $(i', k') \in P \times Q$. Thereby, only one index i corresponds to only one arbitrarily chosen index combination (i', k').

Note, the Fourier–Motzkin elimination of a matrix, $\mathbf{A}$, does not exist iff $Z = P = \emptyset$ or $Z = Q = \emptyset$, i.e. the last column of the matrix $\mathbf{A}$ solely comprises either positive or negative elements.

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