# Designing Communication Topologies for Optimal Synchronization Trajectories of Homogeneous Linear Multi-Agent Systems

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***Abstract*— In this paper the synchronization of homogeneous linear multi-agent systems is considered. In this process, all agents are required to converge to a common trajectory called synchronization trajectory. Without synchronization every single agent would have followed its own autonomous trajectory determined by its initial state. Hence, we interpret the initial states of the agents as their respective preferences. We define a cost function that penalizes the compromise each agent has to make when the synchronization trajectory differs from the agent's preferred trajectory. Using classical synchronization controllers, the synchronization trajectory essentially depends on the communication topology and the initial states of the agents. Therefore, we pose optimization problems concerned with finding the communication topology yielding minimal cost, i.e. the synchronization trajectory that constitutes an optimal compromise for all agents. In this respect, we minimize the cost for a given initial state, for the worst-case as well as for the average over all admissible initial states of the agents. The introduced minimization problems are reformulated as semidefinite programs so that they can be efficiently solved. A numerical example illustrates the results.**

## I. INTRODUCTION

The synchronization of multi-agent systems has been the topic of a considerable amount of literature in the past decade. This is due to its application in various fields as, e.g., multi-vehicle control [1], sensor fusion [2] and synchronization in networks of phase oscillators [3]. Most of the literature concentrates on the design of synchronization controllers for both homogeneous and heterogeneous multi-agent systems, e.g. [4], [5], [6], [7].

In this contribution we will focus on designing an optimal synchronization trajectory in the following sense. Interpreting the initial states of the agents as their respective preferences, we define a cost function that describes the compromise each agent has to make by synchronizing with the others. As an example for such a compromise, one could think of vehicles driving on a highway that synchronize their velocities to form a platoon. Each vehicle prefers to drive at a certain velocity. As a platoon the different vehicles have to find a common velocity. Therefore, every vehicle has to compromise in adjusting its own velocity to the platoon velocity. The cost function to be defined penalizes this compromise.

This means, we deal with the problem of finding a synchronization trajectory that takes the preferences of the different agents into account. When using linear synchronization controllers, the communication topology directly affects the synchronization trajectory. Therefore, we optimize the communication topology underlying the synchronization process. We take the cost function of each agent into account and design the communication topology to minimize the overall cost of the multi-agent system. To achieve this, we give constructive theorems by formulating semidefinite convex programs which can be efficiently solved.

There are several approaches in the literature that consider the design of communication topologies. However, the approaches mainly focus on optimizing the edge weights of the graph describing the topology in order to maximize the algebraic connectivity of the graph. Using results from [8], [9] treats the problem by utilizing the eigenvectors associated with the algebraic connectivity. Boyd casts the problem into a convex optimization problem [10] and considers the sum of the edge weights to be bounded. In [11] the problem is refined by bounding the number of edges which results in a mixed integer semidefinite program. All these approaches relate to the transient synchronization behaviour of a multi-agent system. Our contribution is to show the possibility of using the communication topology to affect the stationary synchronization trajectory in order to implement some kind of hierarchy into the system.

The paper is organized as follows. In Section II we give some preliminaries concerning communication graphs and synchronization of homogeneous multi-agent systems and we define the cost of the resulting synchronization trajectory. Section III provides our main results in formulating different optimization problems and reformulating these problems as semidefinite programs. In Section IV the theoretical results are analyzed using a numerical example. The paper is concluded in Section V.

## II. PRELIMINARIES

### A. Mathematical Notations

We use the following notational conventions. $\mathrm{Re}\{z\}$ and $\mathrm{Im}\{z\}$ denote real and imaginary parts of a complex number $z$, respectively. The $n \times n$ identity matrix is denoted by $\boldsymbol{I}_n$, whereas the zero matrix $\mathbf{0}$ has appropriate dimensions, if not denoted explicitly by $\mathbf{0}_{a\times b} \in \mathbb{R}^{a\times b}$. The $n \times 1$ vector with the $i$-th element equal to one and all other elements equal to zero is written as $\boldsymbol{e}_i$ and the $n\times 1$ vector of all ones is written as $\mathbf{1}_n$. $\|\cdot\|_2$ denotes the Euclidean norm. $\mathbb{R}$ are the real, $\mathbb{Q}$ the rational and $\mathbb{N}$ the natural numbers. $\lambda_i(\boldsymbol{A})$ denotes the $i$-th eigenvalue of a matrix $\boldsymbol{A}$, $\mathrm{tr}\,(\boldsymbol{A})$ its trace, and $|\boldsymbol{A}|$ its determinant. $\boldsymbol{A} \succeq \mathbf{0}$ means that the matrix $\boldsymbol{A}$ is positive semidefinite, whereas $\boldsymbol{A} \geq \mathbf{0}$ means that $\boldsymbol{A}$ is elementwise nonnegative. Let $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\top \in \mathcal{P} \subseteq \mathbb{R}^n$ with

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$\mathcal{P}$ compact and $f : \mathcal{P} \to \mathbb{R}$, then we write $\int_{\mathcal{P}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ to denote the integral of $f(\boldsymbol{x})$ over the set $\mathcal{P}$. By $\otimes$ we denote the Kronecker product. $\mathrm{card}(M)$ denotes the cardinality of a set $M$. The support of a multiset $\Omega$, i.e. the set of unique elements in $\Omega$, is denoted by $\mathrm{supp}(\Omega)$ and for an element $\omega \in \Omega$ the multiplicity is $m_\Omega(\omega)$.

### B. Graphs and Associated Matrices

We use graphs to describe the communication within multi-agent systems. The following definitions and basic properties of graphs are taken from [12]. A graph is formally defined by the pair $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ with the finite vertex set $\mathcal{V}_\mathcal{G} = \{\nu_1, \ldots, \nu_N\}$ and the edge set $\mathcal{E}_\mathcal{G} \subseteq \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$. We consider directed graphs. An edge is therefore an ordered pair of vertices and we write $(\nu_i, \nu_j)$ to denote an edge from vertex $\nu_i$ to vertex $\nu_j$. A path of length $r$ in $\mathcal{G}$ is a sequence $\nu_{k_0}, \ldots, \nu_{k_r}$ of $r+1$ vertices such that $(\nu_{k_{i-1}}, \nu_{k_i}) \in \mathcal{E}_\mathcal{G}$ for all $i \in \{1, \ldots, r\}$. A graph is said to be connected if there exists at least one vertex such that there exists a path from this vertex to any other vertex. The adjacency matrix $\boldsymbol{A}_\mathcal{G}$ of the graph $\mathcal{G}$ is defined as

$$\boldsymbol{A}_\mathcal{G} = [a_{ij}] \begin{cases} > 0, & \text{if } (v_j, v_i) \in \mathcal{E}_\mathcal{G} \\ = 0, & \text{otherwise,} \end{cases}$$

where we allow for positive edge weights, which represent weightings in the communication of the agents. We exclude graphs with self-loops, i.e. it holds $a_{ii} = 0$ for all $i \in \{1, \ldots, N\}$. The degree matrix $\boldsymbol{D}_\mathcal{G}$ is a diagonal matrix with diagonal elements $d_{ii} = \sum_{k=1}^{N} a_{ik},\ i = 1, \ldots, N$. The degree matrix and the adjacency matrix define the graph Laplacian matrix $\boldsymbol{L}_\mathcal{G} = \boldsymbol{D}_\mathcal{G} - \boldsymbol{A}_\mathcal{G}$. The graph is connected if and only if $\mathrm{rank}\,(\boldsymbol{L}_\mathcal{G}) = N - 1$. The right eigenvector corresponding to the zero eigenvalue of $\boldsymbol{L}_\mathcal{G}$ is the all one vector $\boldsymbol{1}_N$. The respective left eigenvector $\boldsymbol{\gamma}^\top \in \mathbb{R}^N$ (whose direction is unique in the case of a connected graph) has solely nonnegative elements and can be chosen such that $\boldsymbol{\gamma}\boldsymbol{1}_N = 1$ (cf. [13]). All nonzero eigenvalues have positive real part. We will assume the eigenvalues of $\boldsymbol{L}_\mathcal{G}$ to be ordered according to their real part, i.e. for a connected graph holds $0 = \mathrm{Re}\,\{\lambda_1(\boldsymbol{L}_\mathcal{G})\} < \mathrm{Re}\,\{\lambda_2(\boldsymbol{L}_\mathcal{G})\} \le \cdots \le \mathrm{Re}\,\{\lambda_N(\boldsymbol{L}_\mathcal{G})\}$.

### C. Synchronization of Homogeneous Linear Multi-Agent Systems

Throughout this paper, we consider a system of $N$ homogeneous linear time-invariant agents

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}\boldsymbol{u}_i, \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i,0}, \tag{1}$$

where $\boldsymbol{x}_i \in \mathbb{R}^n$, $\boldsymbol{u}_i \in \mathbb{R}^m$, and $\boldsymbol{A}$, $\boldsymbol{B}$ are matrices of appropriate dimensions. We use the notation $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_1^\top & \ldots & \boldsymbol{x}_N^\top\end{bmatrix}^\top$ for the stacked state vector of all agents. In the same way, $\boldsymbol{x}_0$ describes the stacked vector of initial states. The agents are communicating over a static directed graph $\mathcal{G}$, which is described by its Laplacian matrix $\boldsymbol{L}_\mathcal{G}$. We make the two basic assumptions:

*Assumption 1:* $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable.

*Assumption 2:* $\mathcal{G}$ is connected.

To synchronize the agents the synchronization protocol with the homogeneous synchronization gain $\boldsymbol{K}$

$$\boldsymbol{u}_i = \boldsymbol{K} \sum_{j=1}^{N} l_{ij} \boldsymbol{x}_j \tag{2}$$

is employed where $l_{ij}$ are the elements of the Laplacian matrix $\boldsymbol{L}_\mathcal{G}$. The goal of designing the synchronization gain $\boldsymbol{K}$ is to achieve state synchronization, i.e.

$$\lim_{t\to\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\|_2 = 0,\ \forall i, j \in \{1, \ldots, N\}.$$

Assumptions 1 and 2 are necessary and sufficient conditions for the existence of such a synchronizing gain $\boldsymbol{K}$ which must place all eigenvalues of the matrices $\boldsymbol{A} - \lambda_i(\boldsymbol{L}_\mathcal{G})\boldsymbol{B}\boldsymbol{K}$, $i = 2, \ldots, N$ in the open left half plane [14] and could be calculated e.g. by using the results in [4]. We make the following additional assumptions which are similarly made in [15].

*Assumption 3:* All eigenvalues $\lambda_i(\boldsymbol{A})$ have nonpositive real part. Furthermore, there exist eigenvalues that satisfy $\mathrm{Re}\,\{\lambda_i(\boldsymbol{A})\} = 0$. These eigenvalues are assumed to be semi-simple.

We define the multiset $\Omega$ which contains the imaginary parts of all eigenvalues of $\boldsymbol{A}$, that lie on the imaginary axis and have nonnegative imaginary part, according to their respective multiplicity.

*Assumption 4:* It holds $\omega_i/\omega_j \in \mathbb{Q}\ \forall \omega_i, \omega_j \in \Omega\backslash\{0\}$.

Assumption 3 guarantees the synchronization trajectory to be nontrivial and bounded, which makes it possible to define a cost for the stationary behavior as will be done in Section II-D. Assumption 4 allows us to define a time period $T$ of the synchronization trajectory which is such that $\forall \omega_i \in \Omega\backslash\{0\}$, $\exists k_i \in \mathbb{N}$ such that $T = k_i \frac{2\pi}{\omega_i}$ holds. Note that, as also mentioned in [15], Assumption 4 is not restrictive since $\mathbb{Q}$ is dense in $\mathbb{R}$. As in [15], we assume without loss of generality that $\boldsymbol{A}$ is in the form

$$\boldsymbol{A} = \mathrm{diag}\left(\boldsymbol{A}_{\mathrm{stab}}, \boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_{N_\Omega}\right) \tag{3}$$

with $N_\Omega = \mathrm{card}(\mathrm{supp}(\Omega)\backslash\{0\})$ and

$$\begin{aligned} \boldsymbol{A}_0 &= \boldsymbol{0}_{m_\Omega(0)\times m_\Omega(0)} \\ \boldsymbol{A}_i &= \omega_i \left(\boldsymbol{I}_{m_\Omega(\omega_i)} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \end{aligned}$$

for $i = 1, \ldots, N_\Omega$, $\omega_i \in \mathrm{supp}(\Omega)\backslash\{0\}$ and $\omega_j \neq \omega_i$ unless $j = i$. Moreover $\boldsymbol{A}_{\mathrm{stab}}$ contains all eigenvalues of $\boldsymbol{A}$ with negative real part and has dimension $n_{\mathrm{stab}}$. We use this rather technical structure of the system matrix $\boldsymbol{A}$ to conveniently describe the cost for the synchronization trajectory in Section II-D.

Let $\boldsymbol{x}_{\mathrm{syn}}(t)$ describe the synchronization trajectory, i.e. the trajectory that all agents converge to as $t \to \infty$ by using the synchronizing input (2). The synchronization trajectory is given by the autonomous system

$$\dot{\boldsymbol{x}}_{\mathrm{syn}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{syn}},\ \boldsymbol{x}_{\mathrm{syn}}(0) = \boldsymbol{x}_{\mathrm{syn},0} \tag{4a}$$

$$\boldsymbol{x}_{\mathrm{syn},0} = \left(\boldsymbol{\gamma}^\top \otimes \boldsymbol{I}_n\right)\boldsymbol{x}_0, \tag{4b}$$

where $\gamma^\top$ describes the left eigenvector of $L_\mathcal{G}$ associated with the zero eigenvalue and is scaled such that $\gamma^\top \mathbf{1}_N = 1$ (cf. [4]).

*Remark 1:* Wieland *et al.* show in [16] that the synchronization of linear multi-agent systems (including heterogeneous systems) with linear synchronization controllers necessarily leads to output trajectories that are described by autonomous linear systems as in (4a). In particular, [16] proposes a common approach where each agent implements a homogeneous virtual exosystem generating such an output trajectory and a controller to exactly track it. Output synchronization is then achieved by synchronizing the homogeneous exosystems using a synchronization protocol such as (2) for which relation (4b) holds. This means that the results concerning the synchronization trajectory developed within this paper can be used for heterogeneous multi-agent systems straight away. This is also true, when the applied methods for exact tracking, which are used in [16], are replaced by optimal tracking methods such as [17] which leads to optimal stationary synchronization [15]. In this case, we can also use the results in [15] to extend the cost function presented in the next section in order to account for, e.g., the required input energy of the agents.

### D. Costs of the Synchronization Trajectory

Equations (4) show that using a static connected communication topology and homogeneous synchronizing control matrices $K$ leads to a synchronization trajectory which solely depends on the initial states of the agents and on the left eigenvector to the zero eigenvalue of the Laplacian matrix $L_\mathcal{G}$ that describes the communication topology. We use this fact to minimize the cost of the synchronization trajectory in several situations. The cost measures the compromise each agent has to make by synchronizing to the others. For this purpose, let $x_i^0(t)$ be the trajectory that agent $i$ would have followed without synchronization, i.e. as an autonomous agent by choosing $u_i = \mathbf{0}$ in (1). Thereby, we define the cost of the synchronization trajectory for each agent as

$$J_{\mathrm{T},i} = \frac{1}{T} \lim_{t_0\to\infty} \int_{t_0}^{t_0+T} \left(x_i(t) - x_i^0(t)\right)^\top Q_i \left(x_i(t) - x_i^0(t)\right) \mathrm{d}t \tag{5}$$

with $Q_i = G_i^\top G_i$ being positive semidefinite. Thus, the cost for agent $i$ is the integral over one time period of the quadratically penalized stationary difference between the synchronization trajectory and the trajectory that agent $i$ would have followed without synchronization. Exploiting the orthogonality of sinusoids similarly as [15] and the references therein, we express (5) equivalently as

$$J_{\mathrm{T},i} = \left(x_{\mathrm{syn}}(0) - x_i(0)\right)^\top \widetilde{G}_i^\top \widetilde{G}_i \left(x_{\mathrm{syn}}(0) - x_i(0)\right), \tag{6}$$

where the positive semidefinite $\widetilde{G}_i^\top \widetilde{G}_i$ is given by

$$\widetilde{G}_i^\top \widetilde{G}_i = \mathrm{diag}\left(\mathbf{0}_{n_\mathrm{stab}\times n_\mathrm{stab}}, G_{i,0}^\top G_{i,0}, \widetilde{G}_{i,1}^\top \widetilde{G}_{i,1}, \ldots, \widetilde{G}_{i,N_{\overline{\Omega}}}^\top \widetilde{G}_{i,N_{\overline{\Omega}}}\right) \tag{7}$$

with

$$\widetilde{G}_{i,j}^\top \widetilde{G}_{i,j} = \frac{1}{2}\left(G_{i,j}^\top G_{i,j} + M_j^\top G_{i,j}^\top G_{i,j} M_j\right). \tag{8}$$

In this expression $G_{i,j}$ are the columns of $G_i$ that correspond to the respective diagonal block $A_j$ of $A$ in (3) and $M_j$ is given by

$$M_j = I_{m_{\overline{\Omega}}(\omega_j)} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since the cost in (5) is defined for $t_0 \to \infty$, the block $A_\mathrm{stab}$, i.e. the stable part of $A$, has no influence on the cost and thus gives the zero block of $\widetilde{G}_i^\top \widetilde{G}_i$ in (7).

For brevity, we write $E_i = e_i^\top \otimes I_n$ with $e_i \in \mathbb{R}^N$ and $\Gamma = \gamma^\top \otimes I_n$ hereafter, in order that we can write $x_{i,0} = x_i(0) = E_i x_0$ and $x_\mathrm{syn}(0) = \Gamma x_0$ respectively, where the latter follows from (4b). Hence, $J_{\mathrm{T},i}$ in (6) is equivalently written as

$$\begin{aligned} J_{\mathrm{T},i} &= x_0^\top \left(\Gamma - E_i\right)^\top \widetilde{G}_i^\top \widetilde{G}_i \left(\Gamma - E_i\right) x_0 \\ &= x_0^\top \begin{bmatrix} I_{Nn} \\ \Gamma \end{bmatrix}^\top \underbrace{\begin{bmatrix} E_i^\top \widetilde{G}_i^\top \widetilde{G}_i E_i & -E_i^\top \widetilde{G}_i^\top \widetilde{G}_i \\ -\widetilde{G}_i^\top \widetilde{G}_i E_i & \widetilde{G}_i^\top \widetilde{G}_i \end{bmatrix}}_{\widehat{G}_i^\top \widehat{G}_i} \underbrace{\begin{bmatrix} I_{Nn} \\ \Gamma \end{bmatrix}}_{\widehat{\Gamma}} x_0. \end{aligned}$$

Thus, the total cost of the multi-agent system is given by

$$J_\mathrm{T}(\gamma, x_0) = \sum_{i=1}^N J_{\mathrm{T},i} = x_0^\top \widehat{\Gamma}^\top \underbrace{\left(\sum_{i=1}^N \widehat{G}_i^\top \widehat{G}_i\right)}_{\widehat{G}^\top \widehat{G}} \widehat{\Gamma} x_0. \tag{9}$$

Let $n_{\widehat{\mathrm{G}}}$ denote the rank of the $(N+1)n \times (N+1)n$ matrix $\widehat{G}^\top \widehat{G}$. Note that the cost $J_{\mathrm{T},i}$ is zero if the initial state of agent $i$ is equal to the initial state of the synchronization trajectory. Therefore, the total cost $J_\mathrm{T}$ is zero if all agents have the same initial state. This shows that $n_{\widehat{\mathrm{G}}} \le Nn$ even if all $\widetilde{G}_i^\top \widetilde{G}_i$ are positive definite. Consequently, we assume $\widehat{G}$ to be a $n_{\widehat{\mathrm{G}}} \times (N+1)n$ matrix hereafter.

## III. Optimization of the Communication Topology

As shown in Section II, the synchronization trajectory and the costs of the synchronization trajectory depend on the initial states of the agents and on the left eigenvector $\gamma$ corresponding to the zero eigenvalue of the graph Laplacian matrix $L_\mathcal{G}$. We utilize this left eigenvector as a design parameter to minimize the total cost of the multi-agent system given by (9). We consider a specifically given initial state in Section III-A. Often, the inital state is not known *a priori*. Therefore, we also regard the worst-case cost (see Section III-B) and the average cost (see Section III-C) of the synchronization trajectory for all admissible initial states (see Section III-B for the definition of the sets of admissible initial states). Before we start with formulating the single minimization problems, we give one possibility to design a graph Laplacian of a connected graph that has the left eigenvector $\gamma$ to its zero eigenvalue. With $\gamma^\top \mathbf{1}_N = 1$ this could be done by

$$L_\mathcal{G} = I_N - \mathbf{1}_N \gamma^\top. \tag{10}$$

Clearly, $\boldsymbol{\gamma}$ is left eigenvector to the zero eigenvalue. Moreover, $\boldsymbol{L}_\mathcal{G}$ is indeed a Laplacian matrix of a connected graph since it has solely nonnegative diagonal and nonpositive off diagonal elements and zero is an algebraically simple eigenvalue. The latter can be seen by the fact that the eigenvalue 1 has geometric multiplicity $N-1$ since the $N-1$ dimensional kernel of $\boldsymbol{\gamma}^\top$ is the right eigenspace of the eigenvalue 1.

*Remark 2:* Note that (10) provides only one possible choice for a graph Laplacian of a connected graph to have $\boldsymbol{\gamma}$ as left eigenvector to its zero eigenvalue. One could instead use, e.g., a slight modification of the results presented in [11] to design graph Laplacians with the respective left eigevector with minimal communication edges which leads to a mixed integer semidefinite program. But since the focus of this paper is the design of the left eigenvector $\boldsymbol{\gamma}$, we only provide the analytically derived matrix (10).

### A. Minimization of the Cost for a Given Initial State

In the sequel, we choose $\boldsymbol{\gamma}$ in order to minimize the resulting cost of the multi-agent system according to (9) in several situations. First, we consider the case, when the initial state of the multi-agent system $\boldsymbol{x}_0$ is known *a priori*.

*Optimization Problem 1:*

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\boldsymbol{0}} \quad & J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)\\ \text{s.t.} \quad & \boldsymbol{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned}$$

We reformulate this optimization problem as a semidefinite program by the following theorem.

*Theorem 1:* Let $\boldsymbol{\gamma}^*$ solve the semidefinite program

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\boldsymbol{0},\,\widehat{J}_\mathrm{T}} \quad & \widehat{J}_\mathrm{T}\\ \text{s.t.} \quad & \begin{bmatrix}\widehat{J}_\mathrm{T} & \boldsymbol{x}_0^\top\widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top\\ \widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}}\boldsymbol{x}_0 & \boldsymbol{I}_{n_{\widehat{\mathrm{G}}}}\end{bmatrix}\succeq\boldsymbol{0},\\ & \boldsymbol{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned}$$

Then $\boldsymbol{\gamma}^*$ also solves Optimization Problem 1.

*Proof:* First, we reformulate the objective of Optimization Problem 1. Instead of searching for $\boldsymbol{\gamma}$ such that $J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)$ is minimized, we search for $\boldsymbol{\gamma}$ such that, the supremum $\widehat{J}_\mathrm{T}$ of $J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)$ is minimized. Since $\widehat{J}_\mathrm{T}-J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)\geq 0$ holds, substituting $J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)$ by (9) and using the Schur complement (see Lemma A.1) directly gives Theorem 1. ■

### B. Minimization of the Worst-Case Cost

As mentioned above, we consider the case when the initial state $\boldsymbol{x}_0$ is not known *a priori*. In this section, we regard the worst-case cost for all admissible initial states. To this end, we give sets using quadratic forms to define the admissible initial states. We consider two cases. We define a set $\mathcal{P}$ using one quadratic form for all agents which is given by

$$\mathcal{P}=\{\boldsymbol{x}_0\,|\,\boldsymbol{x}_0^\top\boldsymbol{P}\boldsymbol{x}_0\leq 1\} \tag{11}$$

with the positive definite matrix $\boldsymbol{P}\in\mathbb{R}^{Nn\times Nn}$. Motivated by the distributed character of a multi-agent system, we define a second set which allows us to specify the admissible initial states for all agents individually. In this regard, we use an individual quadratic form for each agent $i$ with positive definite matrix $\boldsymbol{P}_i\in\mathbb{R}^{n\times n}$ which leads to the $N$ sets

$$\mathcal{P}_i=\{\boldsymbol{x}_{i,0}\,|\,\boldsymbol{x}_{i,0}^\top\boldsymbol{P}_i\boldsymbol{x}_{i,0}\leq 1\}. \tag{12}$$

Considering these sets for the initial states of all single agents, we define the set $\widetilde{\mathcal{P}}$ of admissible initial states of the multi-agent system by

$$\widetilde{\mathcal{P}}=\{\boldsymbol{x}_0\,|\,\boldsymbol{x}_{i,0}\in\mathcal{P}_i,\,\forall i\in\{1,\dots,N\}\}. \tag{13}$$

*1) Minimizing the Worst-Case Cost when $\boldsymbol{x}_0\in\mathcal{P}$:* We start by using the set $\mathcal{P}$ defined by (11) as the set of all admissible initial states. This leads to

*Optimization Problem 2:*

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\boldsymbol{0}} \quad & \max_{\boldsymbol{x}_0} J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)\\ \text{s.t.} \quad & \boldsymbol{x}_0\in\mathcal{P},\\ & \boldsymbol{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned} \tag{14}$$

Again, we are able to reformulate this problem as a semidefinite program which results in the following theorem.

*Theorem 2:* Let $\boldsymbol{\gamma}^*$ solve the semidefinite program

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\boldsymbol{0},\,\widehat{J}_\mathrm{T}} \quad & \widehat{J}_\mathrm{T}\\ \text{s.t.} \quad & \begin{bmatrix}\widehat{J}_\mathrm{T}\boldsymbol{P} & \widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top\\ \widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}} & \boldsymbol{I}_{n_{\widehat{\mathrm{G}}}}\end{bmatrix}\succeq\boldsymbol{0},\\ & \boldsymbol{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned}$$

Then $\boldsymbol{\gamma}^*$ also solves Optimization Problem 2.

*Proof:* Following the proof of Theorem 1, we reformulate Optimization Problem 2 and minimize $\widehat{J}_\mathrm{T}$ subject to the additional constraint

$$J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)\leq\widehat{J}_\mathrm{T},\ \forall\,\boldsymbol{x}_0\in\mathcal{P}. \tag{15}$$

With $J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)$ according to (9) we use the S-procedure (see Lemma A.2) to express (15) as

$$\widehat{J}_\mathrm{T}\boldsymbol{P}-\widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top\widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}}\succeq\boldsymbol{0}.$$

Note that according to Lemma A.3 the S-procedure gives an equivalent expression in this case, since $\mathcal{P}$ is defined by only one quadratic form and there exists $\boldsymbol{x}_0$, e.g. $\boldsymbol{x}_0=\boldsymbol{0}$, such that $1-\boldsymbol{x}_0^\top\boldsymbol{P}\boldsymbol{x}_0>0$. Using the Schur complement finally gives Theorem 2. ■

*2) Minimizing the Worst-Case Cost when $\boldsymbol{x}_0\in\widetilde{\mathcal{P}}$:* Next, we use the set $\widetilde{\mathcal{P}}$ defined by (13) as the set of all admissible initial states, which leads to

*Optimization Problem 3:*

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\boldsymbol{0}} \quad & \max_{\boldsymbol{x}_0} J_\mathrm{T}(\boldsymbol{\gamma},\boldsymbol{x}_0)\\ \text{s.t.} \quad & \boldsymbol{x}_0\in\widetilde{\mathcal{P}},\\ & \boldsymbol{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned}$$

Since we maximize a convex objective under convex constraints in Optimization Problems 2 and 3, these are NP-hard (cf. [18]). This is the reason why the S-procedure as a convexification technique is popular to deal with such problems. However, the S-procedure in the more general case of Optimization Problem 3 is not guaranteed to be lossless ($\widetilde{\mathcal{P}}$ is defined by multiple quadratic forms). Nevertheless, we will again use the S-procedure to handle Optimization Problem 3, so that we are able to provide the following theorem.

*Theorem 3:* Let $\boldsymbol{\gamma}^*$, $\widehat{J}_{\mathrm{T}}^*$ solve the semidefinite program

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\mathbf{0},\,\widehat{J}_{\mathrm{T}},\,\tau_1,\ldots,\tau_N} \quad & \widehat{J}_{\mathrm{T}} \\ \text{s.t.} \quad & \begin{bmatrix} \sum_{i=1}^{N}\tau_i\left(\boldsymbol{e}_i\boldsymbol{e}_i^\top\otimes\boldsymbol{P}_i\right) & \widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top \\ \widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}} & \boldsymbol{I}_{n_{\widehat{\mathrm{G}}}}\end{bmatrix}\succeq\mathbf{0}, \\ & \widehat{J}_{\mathrm{T}}-\sum_{i=1}^{N}\tau_i\geq 0, \\ & \tau_i\geq 0,\ i=1,\ldots,N, \\ & \mathbf{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned}$$

Then $J_{\mathrm{T}}(\boldsymbol{\gamma}^*,\boldsymbol{x}_0)\leq\widehat{J}_{\mathrm{T}}^*$ holds for all $\boldsymbol{x}_0\in\widetilde{\mathcal{P}}$.

*Proof:* Again, consider the reformulation of Optimization Problem 3 to finding the minimal $\widehat{J}_{\mathrm{T}}$ subject to

$$J_{\mathrm{T}}(\boldsymbol{\gamma},\boldsymbol{x}_0)\leq\widehat{J}_{\mathrm{T}},\ \forall\ \boldsymbol{x}_{i,0}\in\mathcal{P}_i,\ i=1,\ldots,N. \tag{16}$$

As in the proof of Theorem 2, we use the S-procedure to deal with (16) which gives

$$\sum_{i=1}^{N}\tau_i\left(\boldsymbol{e}_i\boldsymbol{e}_i^\top\otimes\boldsymbol{P}_i\right)-\widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top\widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}}\succeq\mathbf{0},$$

$$\widehat{J}_{\mathrm{T}}-\sum_{i=1}^{N}\tau_i\geq 0$$

and can be reformulated using the Schur complement to obtain Theorem 3. Since the S-procedure might not be lossless in this more general case, we can only guarantee an upper bound on the worst case in $\widetilde{\mathcal{P}}$. ■

*Remark 3:* We want to emphasize the constructive character of Theorem 3 that provides a choice for $\boldsymbol{\gamma}$, namely $\boldsymbol{\gamma}^*$, that guarantees the worst-case cost to be small although there might be some conservatism due to the relaxation of the problem by the S-procedure.

### C. Minimization of the Average Cost

The optimization of $\boldsymbol{\gamma}$ with respect to the worst case in the last section actually only regards one initial state, namely the worst case. To simultaneously take all admissible initial states into account, we consider the average cost instead of the worst case cost in this section. Again, we use the two sets $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ described in (11) and (13), respectively, to describe the admissible initial states.

*1) Minimizing the Average Cost when $\boldsymbol{x}_0\in\mathcal{P}$:* Consider the case where the initial states are in the set $\mathcal{P}$.

*Optimization Problem 4:*

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\mathbf{0}} \quad & \int_{\mathcal{P}} J_{\mathrm{T}}(\boldsymbol{\gamma},\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_0 \\ \text{s.t.} \quad & \mathbf{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned} \tag{17}$$

The average cost is described by the integral over the set $\mathcal{P}$. Note that the integral could be normalized to obtain the average. We avoid this normalization, since we are only interested in the optimal $\boldsymbol{\gamma}$ which is invariant towards normalization. The optimization problem can be reformulated as a semidefinite program.

*Theorem 4:* Let $\boldsymbol{\gamma}^*$ solve the semidefinite program

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\mathbf{0},\,\mu,\,\boldsymbol{X}} \quad & \mu \\ \text{s.t.} \quad & \begin{bmatrix}\boldsymbol{X} & \boldsymbol{V}'\widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top \\ \widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}}\boldsymbol{V}'^\top & \mu\boldsymbol{I}_{n_{\widehat{\mathrm{G}}}}\end{bmatrix}\succeq\mathbf{0}, \\ & 1-\mathrm{tr}\left(\boldsymbol{X}\right)\geq 0, \\ & \mathbf{1}_N^\top\boldsymbol{\gamma}=1\end{aligned}$$

where $\mu\in\mathbb{R}$ and $\boldsymbol{X}=\boldsymbol{X}^\top\in\mathbb{R}^{Nn\times Nn}$ are slack variables and $\boldsymbol{V}'=\boldsymbol{\Sigma}^{-1/2}\boldsymbol{V}$ with orthonormal $\boldsymbol{V}$ and diagonal $\boldsymbol{\Sigma}$ is given by $\boldsymbol{P}=\boldsymbol{V}^\top\boldsymbol{\Sigma}\boldsymbol{V}$, i.e. the eigenvalue decomposition of $\boldsymbol{P}$. Then $\boldsymbol{\gamma}^*$ also solves Optimization Problem 4.

*Proof:* Consider the coordinate transformation $\boldsymbol{z}_0=\boldsymbol{\Sigma}^{1/2}\boldsymbol{V}\boldsymbol{x}_0$. Within the new coordinates the integral (17) reads

$$\begin{aligned}&\int_{\|\boldsymbol{z}_0\|_2\leq 1}\boldsymbol{z}_0^\top\underbrace{\boldsymbol{\Sigma}^{-1/2}\boldsymbol{V}\widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top\widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}}\boldsymbol{V}^\top\boldsymbol{\Sigma}^{-1/2}}_{\boldsymbol{M}}\boldsymbol{z}_0|\boldsymbol{\Sigma}^{-1/2}|\,\mathrm{d}\boldsymbol{z}_0 \\ &=|\boldsymbol{\Sigma}^{-1/2}|\int_{\|\boldsymbol{z}_0\|_2\leq 1}\sum_{i=1}^{N}\sum_{j=1}^{N}m_{i,j}z_{0,i}z_{0,j}\,\mathrm{d}\boldsymbol{z}_0 \\ &=|\boldsymbol{\Sigma}^{-1/2}|\sum_{i=1}^{N}\sum_{j=1}^{N}m_{i,j}\int_{\|\boldsymbol{z}_0\|_2\leq 1}z_{0,i}z_{0,j}\,\mathrm{d}\boldsymbol{z}_0 \\ &=|\boldsymbol{\Sigma}^{-1/2}|\sum_{i=1}^{N}m_{i,i}\underbrace{\int_{\|\boldsymbol{z}_0\|_2\leq 1}z_{0,i}^2\,\mathrm{d}\boldsymbol{z}}_{f(nN,1)} \\ &=|\boldsymbol{\Sigma}^{-1/2}|f(nN,1)\mathrm{tr}\left(\boldsymbol{M}\right).\end{aligned}$$

As a result, the objective of Optimization Problem 4 is about minimizing the trace of $\boldsymbol{M}$. The trace of $\boldsymbol{M}$ is minimized by introducing the slack variable $\boldsymbol{X}$ (cf. [19, Sec. 2.1]) and by using the Schur complement, which gives Theorem 4. ■

*2) Minimizing the Average Cost when $\boldsymbol{x}_0\in\widetilde{\mathcal{P}}$:* We consider the case when the initial state is in the set $\widetilde{\mathcal{P}}$.

*Optimization Problem 5:*

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\mathbf{0}} \quad & \int_{\mathcal{P}_N}\cdots\int_{\mathcal{P}_1} J_{\mathrm{T}}(\boldsymbol{\gamma},\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_{1,0}\ldots\mathrm{d}\boldsymbol{x}_{N,0} \\ \text{s.t.} \quad & \mathbf{1}_N^\top\boldsymbol{\gamma}=1.\end{aligned} \tag{18}$$

This optimization can be reformulated as a semidefinite program.

*Theorem 5:* Let $\boldsymbol{\gamma}^*$ solve the semidefinite program

$$\begin{aligned}\min_{\boldsymbol{\gamma}\geq\mathbf{0},\,\mu_1,\ldots,\mu_N,\,\boldsymbol{X}_1,\ldots,\boldsymbol{X}_N} \quad & \sum_{i=1}^{N}\mu_i \\ \text{s.t.} \quad & \begin{bmatrix}\boldsymbol{X}_i & \boldsymbol{V}_i'\boldsymbol{E}_i\widehat{\boldsymbol{\Gamma}}^\top\widehat{\boldsymbol{G}}^\top \\ \widehat{\boldsymbol{G}}\widehat{\boldsymbol{\Gamma}}\boldsymbol{E}_i^\top\boldsymbol{V}_i'^\top & \mu_i\boldsymbol{I}_{n_{\widehat{\mathrm{G}}}}\end{bmatrix}\succeq\mathbf{0}, \\ & 1-\mathrm{tr}\left(\boldsymbol{X}_i\right)\geq 0,\ i=1,\ldots,N, \\ & \mathbf{1}_N^\top\boldsymbol{\gamma}=1\end{aligned}$$

where all $\mu_i \in \mathbb{R}$ and $\boldsymbol{X}_i = \boldsymbol{X}_i^\top \in \mathbb{R}^{n\times n}$ are slack variables and $\boldsymbol{V}_i' = \boldsymbol{\Sigma}_i^{-1/2}\boldsymbol{V}_i$ with orthonormal $\boldsymbol{V}_i$ and diagonal $\boldsymbol{\Sigma}_i$ are given by $\boldsymbol{P}_i = \boldsymbol{V}_i^\top\boldsymbol{\Sigma}_i\boldsymbol{V}_i$, i.e. the eigenvalue decomposition of the respective $\boldsymbol{P}_i$. Then $\boldsymbol{\gamma}^*$ also solves Optimization Problem 5.

*Proof:* As in the proof of Theorem 4, we consider the coordinate transformations $\boldsymbol{z}_{i,0} = \boldsymbol{\Sigma}_i^{1/2}\boldsymbol{V}_i\boldsymbol{x}_{i,0}$. Subsequently, we use the abbreviations $\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}} = \widehat{\boldsymbol{G}}\boldsymbol{\Gamma}$ and

$$F = \int_{\mathcal{P}_N} \cdots \int_{\mathcal{P}_1} \mathrm{d}\boldsymbol{x}_{1,0} \ldots \mathrm{d}\boldsymbol{x}_{N,0} = \prod_{i=1}^{N} F_i$$

with

$$F_i = \int_{\mathcal{P}_i} \mathrm{d}\boldsymbol{x}_{i,0} = \left|\boldsymbol{\Sigma}_i^{-1/2}\right| \underbrace{\int_{\|\boldsymbol{z}_i\|_2 \le 1} \mathrm{d}\boldsymbol{z}_i}_{F(n,1)} .$$

We also use the notation

$$F_{\setminus i} = \prod_{\substack{k=1\\k\neq i}}^{N} F_k .$$

With this, the integral (18) in the objective of Optimization Problem 5 reads

$$\begin{aligned}
&\int_{\mathcal{P}_N}\!\!\cdots\!\int_{\mathcal{P}_1}\!\left(\sum_i \boldsymbol{x}_{i,0}^\top\boldsymbol{E}_i\right)\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}^\top\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}\left(\sum_j \boldsymbol{E}_j^\top\boldsymbol{x}_{j,0}\right)\mathrm{d}\boldsymbol{x}_{1,0}\ldots\mathrm{d}\boldsymbol{x}_{N,0}\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\mathcal{P}_N}\!\!\cdots\!\int_{\mathcal{P}_1} \boldsymbol{x}_{i,0}^\top\boldsymbol{E}_i\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}^\top\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}\boldsymbol{E}_j^\top\boldsymbol{x}_{j,0}\,\mathrm{d}\boldsymbol{x}_{1,0}\ldots\mathrm{d}\boldsymbol{x}_{N,0}\\
&= \sum_{i=1}^{N}\int_{\mathcal{P}_N}\!\!\cdots\!\int_{\mathcal{P}_1} \boldsymbol{x}_{i,0}^\top\boldsymbol{E}_i\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}^\top\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}\boldsymbol{E}_i^\top\boldsymbol{x}_{i,0}\,\mathrm{d}\boldsymbol{x}_{1,0}\ldots\mathrm{d}\boldsymbol{x}_{N,0}\\
&= \sum_{i=1}^{N} F_{\setminus i}\int_{\mathcal{P}_i} \boldsymbol{x}_{i,0}^\top\boldsymbol{E}_i\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}^\top\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}\boldsymbol{E}_i^\top\boldsymbol{x}_{i,0}\,\mathrm{d}\boldsymbol{x}_{i,0} \qquad (19)\\
&= \sum_{i=1}^{N} F_{\setminus i} f(n,1)|\boldsymbol{\Sigma}_i^{-1/2}|\,\mathrm{tr}\left(\underbrace{\boldsymbol{V}_i'\boldsymbol{E}_i\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}^\top\widehat{\boldsymbol{G}}_{\boldsymbol{\gamma}}\boldsymbol{E}_i^\top\boldsymbol{V}_i'^\top}_{\boldsymbol{M}_i}\right) \qquad (20)\\
&= \frac{F f(n,1)}{F(n,1)}\sum_{i=1}^{N}\mathrm{tr}\left(\boldsymbol{M}_i\right),
\end{aligned}$$

where (19) $\Rightarrow$ (20) directly follows from the proof of Theorem 4 and adjusting the dimensions (from $Nn$ to $n$). So the objective is to minimize the sum of the traces of the matrices $\boldsymbol{M}_i$. Rewriting the different traces by introducing slack variables $\boldsymbol{X}_i$ and using the Schur complement finally gives Theorem 5. ■

*Remark 4:* From (20) it is easy to see that if the matrix $\boldsymbol{P}$ in the definition of the set $\mathcal{P}$ by (11) is such that $\boldsymbol{P} = \mathrm{diag}\left(\boldsymbol{P}_1, \ldots, \boldsymbol{P}_N\right)$ with $\boldsymbol{P}_i$ from the definition of the sets $\mathcal{P}_i$ by (12) then Optimization Problem 4 and Optimization Problem 5 are equivalent.

TABLE I

RESULTING LEFT EIGENVECTORS WHEN $\boldsymbol{Q}_i$ ARE GIVEN BY (23)

| | $\gamma_1$ | $\gamma_2$ | $\widehat{J}_\mathrm{T}$ | $\overline{J}_\mathrm{T}$ |
|---|---|---|---|---|
| Worst-Case for $\boldsymbol{x}_0 \in \widetilde{\mathcal{P}}$ | 0.5 | 0.5 | 3.125 | 0.691 |
| Average-Case for $\boldsymbol{x}_0 \in \widetilde{\mathcal{P}}$ | 0.615 | 0.385 | 3.291 | 0.654 |
| $\boldsymbol{x}_0 = [0\ \ 2\ \ 0\ \ -0.5]^\top$ | 0.998 | 0.002 | 6.229 | 1.06 |
| $\boldsymbol{x}_0 = [-0.5\ \ 0\ \ 2\ \ 0]^\top$ | 0.5 | 0.5 | 3.125 | 0.691 |

## IV. NUMERICAL EXAMPLE

In this section, we analyze the results of Section III by means of a numerical example which was implemented using [20]. For the purpose of graphical visualization, we keep the example as simple as possible and illustrate some of the effects the choice of the weighting matrices $\boldsymbol{Q}_i$ in the cost function (5) has. Regarding Remark 4, we focus on the set $\widetilde{\mathcal{P}}$. Consider a system with two agents. Let the dynamics of the agents be

$$\dot{\boldsymbol{x}}_i = \boldsymbol{u}_i \qquad (21)$$

with $\boldsymbol{x}_i, \boldsymbol{u}_i \in \mathbb{R}^2$, $i = 1, 2$. This might illustrate the two dimensional platooning of, e.g., mobile robots, coordinating their planar movement. Reminding the motivating discussion in Section I, we look for an optimal compromise on common velocities in both dimensions. We choose

$$\boldsymbol{P}_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0.25 \end{bmatrix},\ \boldsymbol{P}_2 = \begin{bmatrix} 0.25 & 0 \\ 0 & 4 \end{bmatrix},$$

which give the sets of admissible initial states $\mathcal{P}_1$ and $\mathcal{P}_2$ shown in Fig. 1 as blue and red shaded ellipses. In the case of the simple dynamics (21), it holds $\widetilde{\boldsymbol{G}}_i^\top\widetilde{\boldsymbol{G}}_i = \boldsymbol{Q}_i$ according to (7). Thus, by the choice of $\boldsymbol{Q}_i$, we can individually weight the compromise the agents make in the first and the second state by synchronizing to the consensus state. But first, we start with an equivalent weighting of both states so that we choose

$$\boldsymbol{Q}_i = \alpha_i \boldsymbol{I}_2 \qquad (22)$$

with $\alpha_i > 0$. The resulting $\boldsymbol{\gamma}$ for both the worst and the average case are $\gamma_i = \alpha_i/(\alpha_1 + \alpha_2)$, $i = 1, 2$. This is quite intuitive and shows that by making no difference between the weightings of the different states the optimal consensus is directly determined by the weighting of the different agents. Fig. 1 shows the resulting sets of consensus states for different choices of $\alpha_i$. Next, we give an example for weighting the compromise in the two states differently. We choose

$$\boldsymbol{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix},\ \boldsymbol{Q}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.001 \end{bmatrix} \qquad (23)$$

which implies that both agents equally penalize a synchronization compromise in the first state, but agent 1 penalizes a compromise in the second state higher than agent 2. We optimize $\boldsymbol{\gamma}$ according to this weighting. The results of the optimization are shown in Table I. Besides the elements of the optimized $\boldsymbol{\gamma}$, the worst-case cost $\widehat{J}_\mathrm{T}$ and the average cost $\overline{J}_\mathrm{T}$ for the respective $\boldsymbol{\gamma}$ are shown. Four differently optimized

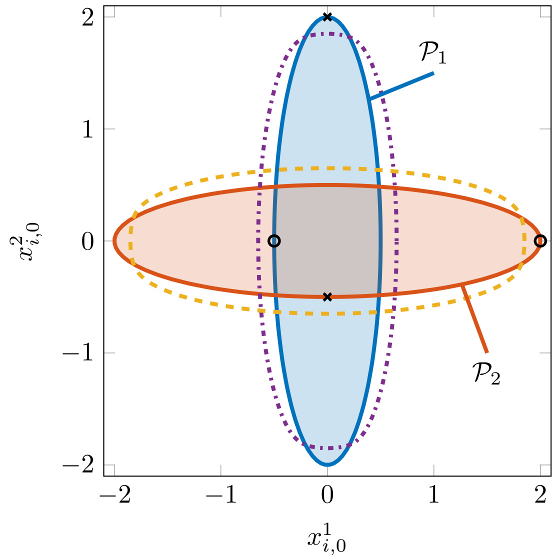


Fig. 1. Sets of admissible initial states $\mathcal{P}_1$ (blue) and $\mathcal{P}_2$ (red). Resulting sets of consensus states for optimized topology with $\boldsymbol{Q}_i$ according to (22) with $\alpha_1 = 0.9$, $\alpha_2 = 0.1$ (dashed) and with $\alpha_1 = 0.1$, $\alpha_2 = 0.9$ (dotted)

$\boldsymbol{\gamma}$ are regarded. The first case shows the left eigenvector that solves the semidefinite program in Theorem 3 regarding the worst-case cost. The second case shows the left eigenvector that minimizes the average cost according to Section III-C.2. It can be seen that the resulting $\boldsymbol{\gamma}$ are different in the two cases. Furthermore, Table I also shows optimal $\boldsymbol{\gamma}$ for two specific initial states, namely $\boldsymbol{x}_0 = \begin{bmatrix} 0 & 2 & 0 & -0.5 \end{bmatrix}^\top$ and $\boldsymbol{x}_0 = \begin{bmatrix} -0.5 & 0 & 2 & 0 \end{bmatrix}^\top$. Optimization with respect to the former (shown as × in Fig. 1) nearly leads to a leader-follower topology with agent 1 as leader. This topology gives a large cost in the worst-case and in the average case. The optimal left eigenvector for the second considered initial state (shown as ∘ in Fig. 1) is the same as the one obtained from the semidefinite program in Theorem 3. Remarkably, that particular $\boldsymbol{x}_0$ solves Optimization Problem 3, which could be verified by numerical means. This shows that in this case Theorem 3 provides the optimal left eigenvector with respect to Optimization Problem 3. Indeed, in the setup considered in this example, the theorem of Polyak applies [21, Thm. 4.1], which is one exceptional case for which the S-procedure is known to be non-conservative.

## V. Conclusion

In this paper, we defined a cost function for the synchronization trajectory of homogeneous linear multi-agent systems. We used the left eigenvector associated with the zero eigenvalue of the graph Laplacian matrix as a design parameter to optimize the communication topology in order to minimize the cost of the synchronization trajectory. Since the defined cost depends on the initial states of the agents, we minimized the cost in different cases: the cost for a given initial state, the worst-case cost and the average cost for all admissible initial states. The respective optimization problems were reformulated as semidefinite programs.

## Appendix

*Lemma A.1 (Schur complement [19, Sec. 2.7.4]):* Suppose $\boldsymbol{Q}$, $\boldsymbol{R}$ are symmetric matrices and $\boldsymbol{R}$ is positive definite. The condition

$$\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^\top & \boldsymbol{R} \end{bmatrix} \succeq \boldsymbol{0}$$

is equivalent to $\boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^\top \succeq \boldsymbol{0}$.

*Lemma A.2 (S-procedure [22]):* Let $F_i(\boldsymbol{x})$, $i = 0, \dots, m$ be real functions on an arbitrary set $\mathcal{X}$. Let $\tau_i$, $i = 1, \dots, m$ be real numbers. The condition

$$F_0(\boldsymbol{x}) \geq 0 \text{ for } F_1(\boldsymbol{x}) \geq 0, \dots, F_m(\boldsymbol{x}) \geq 0 \tag{24}$$

is implied by the condition

$$\exists \tau_i \geq 0 \, : \, F_0(\boldsymbol{x}) - \sum_{i=1}^{m} \tau_i F_i(\boldsymbol{x}) \geq 0, \; \forall \boldsymbol{x} \in \mathcal{X}. \tag{25}$$

*Remark A.1 ([19, Sec. 2.6.3]):* Suppose $\boldsymbol{x} \in \mathbb{R}^n$ and $F_i(\boldsymbol{x})$, $i = 0, \dots, m$ are quadratic functions of the form

$$F_i(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{P}_i \boldsymbol{x} + 2\boldsymbol{u}_i^\top \boldsymbol{x} + v_i, \quad i = 0, \dots, m. \tag{26}$$

Then (25) can be equivalently written as

$$\begin{bmatrix} \boldsymbol{P}_0 & \boldsymbol{u}_0 \\ \boldsymbol{u}_0^\top & v_0 \end{bmatrix} - \sum_{i=1}^{m} \tau_i \begin{bmatrix} \boldsymbol{P}_i & \boldsymbol{u}_i \\ \boldsymbol{u}_i^\top & v_i \end{bmatrix} \succeq \boldsymbol{0}.$$

*Definition A.1 ([22]):* The S-procedure is called lossless, if (24) implies (25), i.e., if (24) and (25) are equivalent.

*Lemma A.3 ([19, Sec. 2.6.3]):* Assume that $\boldsymbol{x} \in \mathbb{R}^n$ and $F_0(\boldsymbol{x})$, $F_1(\boldsymbol{x})$ are quadratic functions of the form (26). Further assume that there exists some $\boldsymbol{x}_0$ such that $F_1(\boldsymbol{x}_0) > 0$. Then the S-procedure for $F_0(\boldsymbol{x})$, $F_1(\boldsymbol{x})$ is lossless.

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