Equilibria of recurrent fuzzy systems☆

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Abstract

Unlike static fuzzy systems, recurrent fuzzy systems are equipped with feedback loops and thus exhibit dynamic behaviors. The dynamics of a recurrent fuzzy system is largely determined by its rule base. The dynamic behavior of a significant subclass of recurrent fuzzy systems may be immediately deduced from their rule base, without need for analyzing their mathematical description. Their equilibrium points may be readily identified and their stability behaviors investigated based on their rule base. The investigations involved lead to convergence theorems and other statements that preclude chaotic dynamics.

Keywords: Recurrent fuzzy systems; Equilibrium point; Fixed point; Stability; Automata; Chaos; Linguistic dynamic

1. Introduction

In general, fuzzy methods are utilized for incorporating knowledge gained from human experience into mathematical descriptions of systems. These systems are usually static. Dynamic systems using fuzzy methods are either variations of Takagi–Sugeno systems [25] or they describe a static function, some of whose input values are time delayed, fed back, output values. While iterated fuzzy sets [12], dynamic fuzzy systems [14–16,22], fuzzy finite state machines [10,20,26], and fuzzy automata [18,27] feed back fuzzy sets or fuzzy values, recurrent fuzzy systems [1–5,9] feed back defuzzified values. Thus, recurrent fuzzy systems are easy to implement and are used in an industrial application [1,2]. The structure of recurrent fuzzy systems was contemporaneously and independently introduced in [1,2,9]. In [1,2], it was motivated by its similarity to automata, and, in [9], by its similarity to recurrent neural nets. In [4], we used the term "recurrent fuzzy system" to cover both aspects of these systems and gave first systematic steps towards a theory of recurrent fuzzy systems. The present

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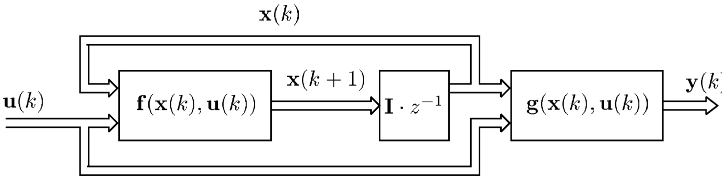

Fig. 1. Block schematic of a general discrete-time system. **I** is the identity matrix.

work provides a further step to such a theory by investigating the equilibrium points of recurrent fuzzy systems defined in [4]. Since recurrent fuzzy systems are also interpretable on a linguistic basis, statements regarding their equilibrium points and stability behaviors will be presented here such that they may be readily derived at this qualitative, linguistic, level.

A recurrent fuzzy system is a time-discrete, nonlinear system, like that shown in Fig. 1, and may be described by a transition function, **f**, an output function, **g**, and the equations

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)), \tag{1}$$

$$\mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)), \tag{2}$$

where **f** and **g** are fuzzy functions covering fuzzification, inference, and defuzzification. For every given input vector, $\mathbf{u}(k) \in \mathbb{R}^m$, the transition function, **f**, maps the crisp state vector, $\mathbf{x}(k) \in \mathbb{R}^n$, at time k onto a new state vector, $\mathbf{x}(k+1) \in \mathbb{R}^n$, at time $k+1$. The output function, **g**, determines an output vector, $\mathbf{y}(k)$, for every input vector, $\mathbf{u}(k)$, and state vector, $\mathbf{x}(k)$. However, **g** has no effect on system dynamics, and will thus receive no further consideration here.

Due to the configuration of recurrent fuzzy systems, which has been described in detail in [4] and will be only briefly covered below, the interrelations involved may be considered at both a qualitative, linguistic level and a quantitative, mathematical level.

At the linguistic level, recurrent fuzzy systems are completely described by a rule base that specifies the new linguistic values, $L_{w_l}^{x_l}$, that their state variables, $x_l(k+1)$, assume for given linguistic values, $L_{j_i}^{x_i}$, of their state variables, $x_i(k)$, and given linguistic values, $L_{q_p}^{u_p}$, of their input parameters, $u_p(k)$. That rule base should be complete, free of contradictions, and contain exclusively rules having the following form:

$$\text{if } \mathbf{x}(k) = \mathbf{L}_{\mathbf{j}}^{\mathbf{x}} \quad \text{and} \quad \mathbf{u}(k) = \mathbf{L}_{\mathbf{q}}^{\mathbf{u}}, \text{ then } \mathbf{x}(k+1) = \mathbf{L}_{\mathbf{w}}^{\mathbf{x}}, \tag{3}$$

where the so-called linguistic vectors, $\mathbf{L}_{\mathbf{j}}^{\mathbf{x}}$, $\mathbf{L}_{\mathbf{q}}^{\mathbf{u}}$, and $\mathbf{L}_{\mathbf{w}}^{\mathbf{x}}$, involved represent the "and" correlation of their respective components, $L_{j_i}^{x_i}$, $L_{u_p}^{q_p}$, and $L_{w_l}^{x_l}$, e.g., $\mathbf{x} = \mathbf{L}_{\mathbf{j}}^{\mathbf{x}} = (L_{j_1}^{x_1}, L_{j_2}^{x_2}, L_{j_3}^{x_3})$ is a short form for "$x_1 = L_{j_1}^{x_1}$ and $x_2 = L_{j_2}^{x_2}$ and $x_3 = L_{j_3}^{x_3}$".

This linguistic representation of a recurrent fuzzy system's dynamics may be formalized even further: the rule base can then be regarded as an operator that maps its linguistic vectors onto new linguistic vectors that, together with its feedback loop, constitutes a finite automaton, which is also termed a "linguistic automaton" [4]. This approach allows employing methods of representation, such as state graphs, and analytical methods taken from automaton theory.

The connection between the qualitative level and the quantitative level is made employing fuzzy methods: every linguistic value, $L_{j_i}^{x_i}$, of a state parameter, x_i, is associated with a convex, membership function, $\mu_{j_i}^{x_i}(x_i)$, that is used to fuzzify the crisp state parameter x_i. The values of these membership functions add up to unity at every point, x_i, i.e.

$$\sum_{j_i} \mu_{j_i}^{x_i}(x_i) = 1. \tag{4}$$

A so-called "core position," $s_{j_i}^{x_i}$, that serves as the singleton position during defuzzification, is selected from the core, $\{x_i | \mu_{j_i}^{x_i}(x_i) = 1\}$, of each membership function, $\mu_{j_i}^{x_i}(x_i)$, such that $\mu_{j_i}^{x_i}(s_{j_i}^{x_i}) = 1$. It can be interpreted as a "typical" numerical value for the linguistic value $L_{j_i}^{x_i}$. The core positions, $s_{j_i}^{x_i}$, can be combined forming a core position vector, $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$.

A similar rule applies to every input parameter, u_p, and their linguistic values, $L_{q_p}^{u_p}$. So, we obtain membership functions, $\mu_{q_p}^{u_p}(u_p)$, and associated core positions, $s_{q_p}^{u_p}$, with $\mu_{q_p}^{u_p}(s_{q_p}^{u_p}) = 1$.

If the membership functions, $\mu_{j_i}^{x_i}(x_i)$ and $\mu_{q_p}^{u_p}(u_p)$, are then employed for fuzzification, the rule base of Eq. (3), together with employment of PROD-SUM-operators [8] for the inference and the core positions, $s_{j_i}^{x_i}$, and the CoS-method [8] for defuzzification, will then determine the transition function, $\mathbf{f}$, which has the following analytical form:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \sum_{\mathbf{j},\mathbf{q}} \mathbf{s}_{\mathbf{w}(\mathbf{j},\mathbf{q})}^{\mathbf{x}} \prod_i \mu_{j_i}^{x_i}(x_i) \prod_p \mu_{q_p}^{u_p}(u_p), \tag{5}$$

where the core position vectors, $\mathbf{s}_{\mathbf{w}(\mathbf{j},\mathbf{q})}^{\mathbf{x}}$, are those associated with the linguistic vectors, $\mathbf{L}_{\mathbf{w}}^{\mathbf{x}}$, of $\mathbf{x}(k+1)$ appearing in the rule base (3) and their index vectors, $\mathbf{w}$, will thus be uniquely determined by the respective index vectors, $\mathbf{j}$ and $\mathbf{q}$, of the linguistic variables, $\mathbf{L}_{\mathbf{j}}^{\mathbf{x}}$, of $\mathbf{x}(k)$ and, $\mathbf{L}_{\mathbf{q}}^{\mathbf{u}}$, of $\mathbf{u}(k)$.

Eq. (5) forms the mathematical appendage to the linguistic rule base of Eq. (3), from which the rule base may be immediately derived, which involves computing the value of the function $\mathbf{f}(\mathbf{x}, \mathbf{u})$ for core position vectors $\mathbf{x} = \mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$ and $\mathbf{u} = \mathbf{s}_{\mathbf{q}}^{\mathbf{u}}$. Virtually all of the summations appearing in Eq. (5) will then be zero, leaving just the core position vector, $\mathbf{s}_{\mathbf{w}(\mathbf{j},\mathbf{q})}^{\mathbf{x}}$. Expressed in linguistic terms, the core position vectors satisfy the following equation, which is analogous to Eq. (3):

$$\text{if } \mathbf{x}(k) = \mathbf{s}_{\mathbf{j}}^{\mathbf{x}} \quad \text{and} \quad \mathbf{u}(k) = \mathbf{s}_{\mathbf{q}}^{\mathbf{u}}, \text{ then } \mathbf{x}(k+1) = \mathbf{f}(\mathbf{s}_{\mathbf{j}}^{\mathbf{x}}, \mathbf{s}_{\mathbf{q}}^{\mathbf{u}}) = \mathbf{s}_{\mathbf{w}(\mathbf{j},\mathbf{q})}^{\mathbf{x}}. \tag{6}$$

This correspondence immediately yields the equivalence of linguistic automata and recurrent fuzzy systems, provided that the latter are confined to core position vectors. This close correlation between recurrent fuzzy systems at the quantitative level and linguistic automata at the qualitative level is schematically depicted in Fig. 2 and described in greater detail in [4]. That correlation also yields implicit representations of the transition function, $\mathbf{f}$, which are summarized in Lemma 2 appearing in Appendix A.

If only the linguistic automata of a recurrent fuzzy system are considered, then information regarding the particular choices of membership functions and core positions will be lacking. However, that may have a decisive effect on the distribution of equilibrium points and their stability behaviors, as will be shown in the next section.

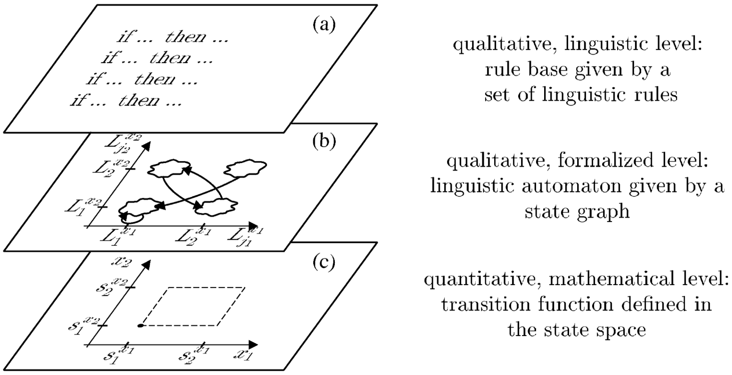

Fig. 2. Description of recurrent fuzzy systems on a qualitative and quantitative level. The wavy nodes of the state graph symbolize the fuzzy nature of the linguistic states.

2. Standardized, recurrent fuzzy systems

Once a rule base has been chosen, completing a recurrent fuzzy system involves the further step of specifying the membership functions and core positions for its state parameters, x_i, and its input parameters, u_p. In the case of static fuzzy systems, the type of membership functions will, in general, not be a matter of major importance. Switching to employing other types of membership functions will usually yield only minor variations in their static input/output behavior. However, in the case of dynamic systems, and, particularly in the case of recurrent fuzzy systems, those minor variations may significantly affect their dynamic behavior.

That effect may be clearly illustrated, based on the example of the following, very simple, recurrent fuzzy system whose rule base is described in terms of a "verbally identical mapping":

$$\text{If } x(k) = \textit{small}, \quad \text{then } x(k+1) = \textit{small}. \tag{7}$$

$$\text{If } x(k) = \textit{large}, \quad \text{then } x(k+1) = \textit{large}. \tag{8}$$

The rule base dictates the system not to change its state.

One would expect that a recurrent fuzzy system based on those rules would also represent a mathematically identical mapping. For the purpose of our investigations, its core positions have been set to $s_1^x = 0$ for $L_1^x = \textit{small}$ and $s_2^x = 1$ for $L_2^x = \textit{large}$. Various types of functions will now be chosen as membership functions, where specifying $\mu_2^x(x)$ will alone be sufficient, since the other will be determined on the interval $[0,1]$ from $\mu_1^x(x) = 1 - \mu_2^x(x)$, in view of Eq. (4). From Eq. (5), the transition function will be given by $f(x) = \mu_2^x(x)$, due to the particular choices of the core positions that have been made.

Fig. 3 depicts four transition functions that result from applying the flank of a triangular function, ($\mu_2^x(x) = x$), a sinusoidal function, ($\mu_2^x(x) = \sin^2(\pi/2 \cdot x)$), and parabolae, ($\mu_2^x(x) = x^2$ and $\mu_2^x(x) = 2x - x^2$), on the interval $[0,1]$, including their effects on the starting points, x, as the total number of iterations, k, of Eq. (1) becomes unbounded.

The fixed points of those mappings may be readily determined as the intersections with the angular bisectors, ($f(x) = x$). In the case of the triangular functions, all starting points are also fixed

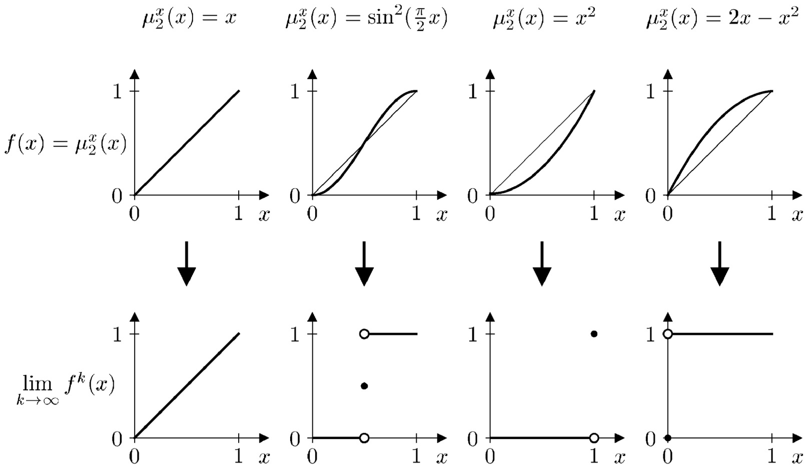

Fig. 3. Various transition functions, $f(x)$, for a linguistical identity mapping and their behavior for $k \to \infty$.

points. Choosing the sinusoidal membership function yields an unstable fixed point at $x = 0.5$ and asymptotically stable fixed points at $x = 0$ and 1. The effects that particular choices of membership functions can have is particularly evident in the case of the parabolae: if $\mu_2^x(x) = f(x) = x^2$, all initial states, $x(0)$, with the exception of $x(0) = 1$, will tend toward $x = 0$, which corresponds to the linguistic value "small." This will even occur for an initial state of, e.g., $x(0) = 0.999$, very close to $x = 1$, which corresponds to the linguistic value "large," which is not evident when considering solely the rule base. The identical mapping will result at the quantitative level as well, as would be expected from the rule base, only if the triangular function is chosen as the membership function. This statement is confirmed by Lemma 3, which appears in Appendix A.

As we have seen, latent dynamics of recurrent fuzzy systems that are not evident from their rule base occur for many types of membership functions. Thus drawing conclusions regarding the dynamic behaviors of recurrent fuzzy systems from the qualitative level to the quantitative level is more difficult, or impossible, in cases involving membership functions other than triangular functions. In order to avoid such latent dynamics, we shall treat here a subclass of recurrent fuzzy systems: standardized, recurrent fuzzy systems, which are defined as follows:

Definition 1. A recurrent fuzzy system will be termed "standardized" if equidistant singleton positions are employed in defuzzifying its state variables and equidistant triangular functions are employed as membership functions in fuzzifying its state variables.

The choice of equidistant singleton positions, i.e. core positions, in Definition 1 does not constitute a severe restriction. A recurrent fuzzy system having nonequidistant core positions may be transformed into a recurrent fuzzy system having equidistant core positions, in which case, one would work internally with equidistant core positions mapped onto its original, nonequidistant, core positions by an output function, **g**. Although such transformations do not, strictly speaking, represent equivalence transformations, both the rule base and the ranges of values involved will agree with one another.

As will be shown in the next section, in the case of standardized, recurrent fuzzy systems, one arrives at far-reaching statements regarding their equilibrium points and stabilities that cannot, in general, be arrived at in the case of fuzzy systems having other types of membership functions.

3. Equilibrium points and nonexpansion

In this section, we shall treat the fundamental aspects of investigating the equilibrium points of recurrent fuzzy systems. We shall treat continuous, recurrent fuzzy systems first and then we shall make use of the special features of the structures of standardized, recurrent fuzzy systems.

By definition, the transition function, $\mathbf{f}$, is a bounded function that maps a system's state space, X, into itself. X contains all vectors, $\mathbf{x}$, whose components, x_i, lie between the lower and upper bounds on the core positions of their respective components, i. Those vectors form a compact, i.e., bounded and closed, subset in the state space. If a component of a state vector, $\mathbf{x}$, lies outside one of those bounds, e.g., if $x_1 < s_1^{x_1}$, then the value of this bounded membership function will be unity, i.e., in this example $\mu_1^{x_1}(x_1) = 1$, which will have the same effect on the fuzzy mechanism as will occur if the component involved lies precisely on that bound, i.e., in this example, if $x_1 = s_1^{x_1}$. Thus, in considering the mapping $\mathbf{f}(X)$, whether the state space, X, is compact or not will be of no consequence. In conjunction with the further considerations involved, it may thus be assumed that the state space, X, is compact.

In the case of continuous, recurrent fuzzy systems (cf. [4]), for every input vector, $\mathbf{u}$, their transition function is a continuous mapping of their state space, X, onto a compact subset of X. Thus according to Brouwer's fixed-point theorem [23], for every input vector, $\mathbf{u}$, the transition function, $\mathbf{f}$, of a continuous, recurrent fuzzy system will have at least one fixed point in its state space, X.

By definition, a fixed point, $\mathbf{x}$, will satisfy $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{x}$ and simultaneously be an equilibrium point of the recurrent fuzzy system, since, we will then have $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) = \mathbf{x}(k)$. We then obtain the following existence theorem for equilibrium points:

Theorem 1. *In a continuous recurrent fuzzy system at least one equilibrium point exists for any input vector*, $\mathbf{u}$.

It would be desirable to know where this equilibrium point, or these equilibrium points, are located. Some equilibrium points may be immediately read off from the rule base: If for a linguistic input vector, $\mathbf{L_q^u}$, there is a linguistic state vector, $\mathbf{L_j^x}$, that is mapped onto itself by the associated rule of the rule base, then that linguistic state vector will henceforth also be termed an "end node" of that input vector. Since that linguistic state vector will remain unaltered when the rule involved is executed and, according to Eqs. (3) and (6), the effects of the rules involved may be immediately transferred to the core positions, it follows that:

Lemma 1. *If a linguistic state vector*, $\mathbf{L_j^x}$, *is an end node for a given linguistic input vector*, $\mathbf{L_q^u}$, *then the state vector* $\mathbf{s_j^x}$ *is an equilibrium point of the recurrent fuzzy system for the input vector* $\mathbf{s_q^u}$.

Statements regarding equilibrium points other than those that lie on core position vectors are, in general, more difficult to arrive at, since several rules will invariably be responsible for that

L_q^u \ L_j^x	1	2	3	4
a	2	3	4	4
b	2	2	3	3
c	1	2	3	2
d	1	1	2	3

Fig. 4. An example for a rule matrix of a rule-continuous recurrent fuzzy system with linguistic values, L_q^u, for its single input variable, u, and linguistic values, L_j^x, for its single state variable, x.

mapping in such cases. Arriving at such statements is simplest, but nontrivial, in the case of recurrent fuzzy systems, which satisfy the following condition: every pair of adjacent rules in their rule matrix has the same, or adjacent, linguistic values in their conclusion, i.e., if the linguistic value of one linguistic variable occurring in the premises is changed to the next-smaller or next-larger linguistic value, then no more than one linguistic value appearing in the conclusion will change, and it will change to either the next-smaller or next-larger value only. A recurrent fuzzy system satisfying this condition is "rule-continuous" [4], since its rule base has a "linguistic continuity," i.e., a continuous structure. An example for a rule matrix of a rule-continuous recurrent fuzzy system with one input variable, u, and one state variable, x, is shown in Fig. 4. The linguistic values of u are $L_1^u =$ "a", $L_2^u =$ "b", $L_3^u =$ "c", $L_4^u =$ "d" and the linguistic values of x are $L_1^x =$ "1", $L_2^x =$ "2", $L_3^x =$ "3", $L_4^x =$ "4". The elements of the rule matrix determine the new linguistic value of the state variable.

In the case of rule-continuous, fuzzy systems, the "distance" between two linguistic state vectors will not be increased when the rules are executed. In order to be able to transfer this property to the quantitative level as well, we will, in the following, consider the case of rule-continuous, standardized, recurrent fuzzy systems. In such systems, for every component, x_i, of their state vector, $\mathbf{x}$, adjacent core positions will have a fixed distance that, in the following, will be designated by Δs_i, i.e., $\Delta s_i = |s_{r+1}^{x_i} - s_r^{x_i}|$. The relation

$$\|\mathbf{x}\| = \sum_i \frac{1}{\Delta s_i}|x_i| \tag{9}$$

then defines a norm within the state space, X, that will, in the following, be termed a "canonical vector norm," and represents a modified Hölder norm (cf. [17], where $p = 1$).

That canonical vector norm allows concluding that the distance between any two, arbitrarily chosen, state vectors, $\mathbf{a}, \mathbf{b} \in X$, under the transition function, $\mathbf{f}$, of a rule-continuous, standardized, recurrent fuzzy system will not increase, i.e., $\|\mathbf{f}(\mathbf{a},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant \|\mathbf{a} - \mathbf{b}\|$, for all $\mathbf{a}, \mathbf{b} \in X$ and all $\mathbf{u} \in U$. In this case, the function $\mathbf{f}(\mathbf{x},\mathbf{u})$ is called nonexpansive [23] in the state vector $\mathbf{x}$ and we have that

Theorem 2. *Rule-continuous, standardized, recurrent fuzzy systems are nonexpansive with respect to their canonical vector norm.*

Proof. The assumption that $\|\mathbf{f}(\mathbf{a},\mathbf{u})-\mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant \|\mathbf{a}-\mathbf{b}\|$ holds for all $\mathbf{a},\mathbf{b}\in X$ and all $\mathbf{u}\in U$ will be proven in three steps:

Step 1: In this step, assume that **a** and **b** differ in no more than one component, i.e., $a_i = b_i$ for $i \neq h$, and that both values a_h and b_h lie in the interval, $[s_r^{x_h}, s_{r+1}^{x_h}]$, which is limited by two adjacent core positions $s_r^{x_h}$ and $s_{r+1}^{x_h}$.

Using $f_l(\mathbf{x},\mathbf{u}) = \sum_{\mathbf{j},\mathbf{q}} s^{x_l}_{w_l(\mathbf{j},\mathbf{q})} \prod_i \mu^{x_i}_{j_i}(x_i) \prod_p \mu^{u_p}_{q_p}(u_p)$ from Eq. (5), it follows that

$$\|\mathbf{f}(\mathbf{a},\mathbf{u})-\mathbf{f}(\mathbf{b},\mathbf{u})\| = \sum_l \frac{1}{\Delta s_l}\left| \sum_{\mathbf{j},\mathbf{q}} s^{x_l}_{w_l(\mathbf{j},\mathbf{q})} \prod_i \mu^{x_i}_{j_i}(a_i) \prod_p \mu^{u_p}_{q_p}(u_p) - \sum_{\mathbf{j},\mathbf{q}} s^{x_l}_{w_l(\mathbf{j},\mathbf{q})} \prod_i \mu^{x_i}_{j_i}(b_i) \prod_p \mu^{u_p}_{q_p}(u_p) \right|. \tag{10}$$

Since $a_i = b_i$ for $i \neq h$ in this case, we get that

$$\|\mathbf{f}(\mathbf{a},\mathbf{u})-\mathbf{f}(\mathbf{b},\mathbf{u})\| = \sum_l \frac{1}{\Delta s_l}\left| \sum_{\mathbf{j},\mathbf{q}} s^{x_l}_{w_l(\mathbf{j},\mathbf{q})} \cdot (\mu^{x_h}_{j_h}(a_h) - \mu^{x_h}_{j_h}(b_h)) \cdot \prod_{i\neq h} \mu^{x_i}_{j_i}(a_i) \prod_p \mu^{u_p}_{q_p}(u_p) \right|. \tag{11}$$

Since the values a_h and b_h of the state vectors lie in the interval $[s_r^{x_h}, s_{r+1}^{x_h}]$, we obtain by employing an auxiliary variable $\Delta\mu = \mu_r^{x_h}(a_h) - \mu_r^{x_h}(b_h)$:

$$\mu^{x_h}_{r+1}(a_h) - \mu^{x_h}_{r+1}(b_h) = (1-\mu^{x_h}_r(a_h)) - (1-\mu^{x_h}_r(b_h)) = -(\mu^{x_h}_r(a_h) - \mu^{x_h}_r(b_h)) = -\Delta\mu \tag{12}$$

and therefore

$$\mu^{x_h}_{j_h}(a_h) - \mu^{x_h}_{j_h}(b_h) = \begin{cases} \Delta\mu & \text{for } j_h = r, \\ -\Delta\mu & \text{for } j_h = r+1, \\ 0 & \text{for } j_h \notin \{r, r+1\}. \end{cases} \tag{13}$$

When we compute the sum by the index j_h in Eq. (11), we obtain:

$$\|\mathbf{f}(\mathbf{a},\mathbf{u})-\mathbf{f}(\mathbf{b},\mathbf{u})\| = \sum_l \frac{1}{\Delta s_l}\left| \sum_{j_i\neq j_h,\mathbf{q}} (s^{x_l}_{w_l(j_1,\ldots,r,\ldots,j_n,\mathbf{q})} - s^{x_l}_{w_l(j_1,\ldots,r+1,\ldots,j_n,\mathbf{q})})\,\Delta\mu \prod_{i\neq h} \mu^{x_i}_{j_i}(a_i) \prod_p \mu^{u_p}_{q_p}(u_p) \right|. \tag{14}$$

When we factor out $\Delta\mu$ and employ the triangular inequality, we can majorize the term given above by

$$\|\mathbf{f}(\mathbf{a},\mathbf{u})-\mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant |\Delta\mu| \sum_l \frac{1}{\Delta s_l}\left(\sum_{j_i\neq j_h,\mathbf{q}} |(s^{x_l}_{w_l(j_1,\ldots,r,\ldots,j_n,\mathbf{q})} - s^{x_l}_{w_l(j_1,\ldots,r+1,\ldots,j_n,\mathbf{q})})| \times \prod_{i\neq h} \mu^{x_i}_{j_i}(a_i) \prod_p \mu^{u_p}_{q_p}(u_p) \right). \tag{15}$$

Since the recurrent fuzzy system is rule-continuous, $s^{x_l}_{w_l(j_1,\dots,r,\dots,j_n,\mathbf{q})}$ and $s^{x_l}_{w_l(j_1,\dots,r+1,\dots,j_n,\mathbf{q})}$ are adjacent or identical core positions and $\sum_L 1/\Delta s_l |s^{x_l}_{w_l(j_1,\dots,r,\dots,j_n,\mathbf{q})} - s^{x_l}_{w_l(j_1,\dots,r+1,\dots,j_n,\mathbf{q})}| \leqslant 1$. Using this inequality, we obtain:

$$\|\mathbf{f}(\mathbf{a},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant |\Delta\mu| \cdot \sum_{j_i \neq j_h, \mathbf{q}} \prod_{i \neq h} \mu^{x_i}_{j_i}(a_i) \prod_p \mu^{u_p}_{q_p}(u_p). \tag{16}$$

When we use the normalizing condition given in Eq. (4) and multiply out the following term, we get

$$1 = \prod_{i \neq h} \left(\sum_{j_i} \mu^{x_i}_{j_i}(x_i) \right) \prod_p \left(\sum_{p_q} \mu^{u_p}_{q_p}(u_p) \right) = \sum_{j_i \neq j_h, \mathbf{q}} \prod_{i \neq h} \mu^{x_i}_{j_i}(a_i) \prod_p \mu^{u_p}_{q_p}(u_p). \tag{17}$$

Thus the sum in Eq. (16) is unity and we have

$$\|\mathbf{f}(\mathbf{a},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant |\Delta\mu|. \tag{18}$$

Since we employ triangular membership functions, we have $|\Delta\mu| = |\mu^{x_h}_r(a_h) - \mu^{x_h}_r(b_h)| = |(1 - 1/\Delta s_h (a_h - s^{x_h}_r)) - (1 - 1/\Delta s_h(b_h - s^{x_h}_r))| = 1/\Delta s_h |a_h - b_h|$ and, in addition, $\|\mathbf{a} - \mathbf{b}\| = \sum_i 1/\Delta s_i |a_i - b_i| = 1/\Delta s_h |a_h - b_h|$. Thus, we obtain in this special case

$$\|\mathbf{f}(\mathbf{a},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant \|\mathbf{a} - \mathbf{b}\|. \tag{19}$$

Step 2: We now require only that **a** and **b** differ in no more that one component, i.e. $a_i = b_i$ for $i \neq h$. But we no longer require that $a_h, b_h \in [s^{x_h}_r, s^{x_h}_{r+1}]$, as we did in step 1.

Without loss of generality, it may be assumed that $a_h \leqslant b_h$. Then a number of core positions, $s^{x_h}_{r+1}, s^{x_h}_{r+2}, \dots, s^{x_h}_{r+t}$, may lie between a_h and b_h. For all $v = 1,\dots,t$, define the components of vectors $\mathbf{d}(v)$ by $d_i(v) = a_i (= b_i)$ for $i \neq h$ and $d_h(v) = s^{x_h}_{r+v}$. Set $\mathbf{d}(0) = \mathbf{a}$ and $\mathbf{d}(t+1) = \mathbf{b}$.

We then have that

$$\begin{aligned} \|\mathbf{f}(\mathbf{a},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| &= \|\mathbf{f}(\mathbf{d}(0),\mathbf{u}) - \mathbf{f}(\mathbf{d}(t+1),\mathbf{u})\| \\ &= \left\| \sum_{v=0}^{t} (\mathbf{f}(\mathbf{d}(v),\mathbf{u}) - \mathbf{f}(\mathbf{d}(v+1),\mathbf{u})) \right\| \\ &\leqslant \sum_{v=0}^{t} \|\mathbf{f}(\mathbf{d}(v),\mathbf{u}) - \mathbf{f}(\mathbf{d}(v+1),\mathbf{u})\|. \end{aligned} \tag{20}$$

Since no further core positions lie between $d_h(v)$ and $d_h(v+1)$ for any v, we can use the result of step 1 to conclude that

$$\begin{aligned} \sum_{v=0}^{t} \|\mathbf{f}(\mathbf{d}(v),\mathbf{u}) - \mathbf{f}(\mathbf{d}(v+1),\mathbf{u})\| &\leqslant \sum_{v=0}^{t} \|\mathbf{d}(v) - \mathbf{d}(v+1)\| \\ &= \sum_{v=0}^{t} \frac{1}{\Delta s_h} |d_h(v) - d_h(v+1)|. \end{aligned} \tag{21}$$

Since $d_h(v)$ increases monotonically with increasing v, we obtain

$$\sum_{v=0}^{t} \frac{1}{\Delta s_h}|d_h(v) - d_h(v+1)| = \frac{1}{\Delta s_h}|d_h(0) - d_h(t+1)| = \frac{1}{\Delta s_h}|a_h - b_h| = \|\mathbf{a} - \mathbf{b}\|. \tag{22}$$

From the chain inequality (20)–(22) it follows that $\mathbf{f}$ is also nonexpansive in the case investigated in step 2.

Step 3: Now $\mathbf{a}$ and $\mathbf{b}$ can be chosen arbitrarily.

Defining vectors $\mathbf{c}(1)$ through $\mathbf{c}(n+1)$ by $c_i(v) = a_i$ for $i \geqslant v$ and $c_i(v) = b_i$ for $i < v$, we have $\mathbf{c}(1) = \mathbf{a}$ and $\mathbf{c}(n+1) = \mathbf{b}$. Then we have that

$$\begin{aligned}\|\mathbf{f}(\mathbf{a},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| &= \|\mathbf{f}(\mathbf{c}(1),\mathbf{u}) - \mathbf{f}(\mathbf{c}(n+1),\mathbf{u})\| \\ &= \left\|\sum_{v=1}^{n}(\mathbf{f}(\mathbf{c}(v),\mathbf{u}) - \mathbf{f}(\mathbf{c}(v+1),\mathbf{u}))\right\| \\ &\leqslant \sum_{v=1}^{n} \|\mathbf{f}(\mathbf{c}(v),\mathbf{u}) - \mathbf{f}(\mathbf{c}(v+1),\mathbf{u})\|. \end{aligned} \tag{23}$$

Since two consecutive vectors, $\mathbf{c}(v)$ and $\mathbf{c}(v+1)$, differ only in their vth component, it follows from step 2:

$$\begin{aligned}\sum_{v=1}^{n} \|\mathbf{f}(\mathbf{c}(v),\mathbf{u}) - \mathbf{f}(\mathbf{c}(v+1),\mathbf{u})\| &\leqslant \sum_{v=1}^{n} \|\mathbf{c}(v) - \mathbf{c}(v+1)\| \\ &= \sum_{v=1}^{n} \frac{1}{\Delta s_v}|c_v(v) - c_v(v+1)| \\ &= \sum_{v=1}^{n} \frac{1}{\Delta s_v}|a_v - b_v| = \|\mathbf{a} - \mathbf{b}\|. \end{aligned} \tag{24}$$

Therefore, $\|\mathbf{f}(\mathbf{a},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant \|\mathbf{a} - \mathbf{b}\|$ holds for any arbitrary choice of vectors, $\mathbf{a}$ and $\mathbf{b}$, from X and the function $\mathbf{f}$ is nonexpansive. □

It should be noted that this theorem remains valid regardless of the choice of the particular input vector, $\mathbf{u}$, which implies that, for any arbitrary sequence of input vectors, $\mathbf{u}(k)$, the distance between two state vectors cannot increase during the course of the mapping in such systems.

The nonexpansion property of Theorem 2 also directly affects the stabilities of equilibrium points provided that a given, fixed, input vector, $\mathbf{u}$, is chosen. We then have that

Theorem 3. *Every equilibrium point in a rule-continuous, standardized, recurrent fuzzy system is stable in the sense of Liapunov.*[1] *For a given equilibrium point*, $\mathbf{b}$, *a Liapunov function is*

[1] Definition and theorems concerning the stability theory of Liapunov can be found in [13,21].

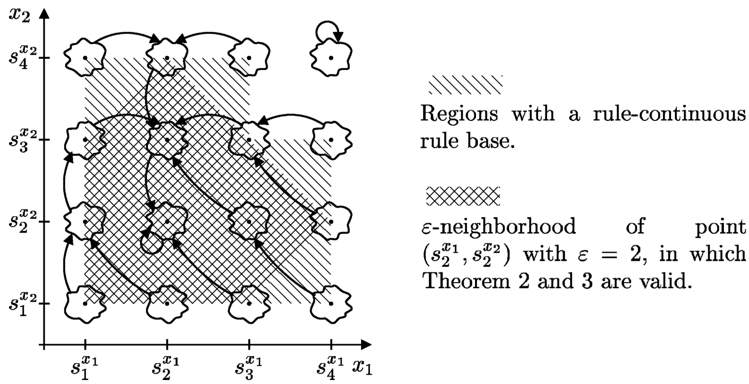

Fig. 5. Theorems 2 and 3 are also applicable in regions of the state space, in which the premise of these theorems are fulfilled.

given by

$$V_{\mathbf{b}}(\mathbf{x}) = \|\mathbf{x} - \mathbf{b}\| = \sum_i \frac{1}{\Delta s_i}|x_i - b_i|. \tag{25}$$

Proof. We have $V_{\mathbf{b}}(\mathbf{b}) = 0$ and $V_{\mathbf{b}}(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{b}$. According to Theorem 2 we have:

$$V_{\mathbf{b}}(\mathbf{f}(\mathbf{x},\mathbf{u})) = \|\mathbf{f}(\mathbf{x},\mathbf{u}) - \mathbf{b}\| = \|\mathbf{f}(\mathbf{x},\mathbf{u}) - \mathbf{f}(\mathbf{b},\mathbf{u})\| \leqslant \|\mathbf{x} - \mathbf{b}\| = V_{\mathbf{b}}(\mathbf{x}). \tag{26}$$

Therefore $\Delta V_{\mathbf{b}} = V_{\mathbf{b}}(\mathbf{f}(\mathbf{x},\mathbf{u})) - V_{\mathbf{b}}(\mathbf{x}) \leqslant 0$. Thus $V_{\mathbf{b}}(\mathbf{x})$ is a Liapunov function and the equilibrium point, $\mathbf{b}$, is stable in the sense of Liapunov. □

Theorems 2 and 3 may also be applied to subregions of state space if the rule base involved is only partially rule-continuous. Fig. 5 illustrates this in terms of an example. Shown there is a two-dimensional state space, with dots marking the core position vectors. Superimposed on that state space is the state graph of a linguistic automaton, which is represented by the wavy nodes and transition arrows. The rule base is rule-continuous over the hatched region, as may readily be verified by referring to the transition arrows. The crosshatched region indicates a ε-neighborhood for which $\varepsilon = 2$ about the equilibrium point, $(s_2^{x_1}, s_2^{x_2})$, which lies entirely within the hatched region, and over which the transition function is nonexpansive. The equilibrium point, $(s_2^{x_1}, s_2^{x_2})$, is stable in the sense of Liapunov.

This type of stability of an equilibrium point is no guarantee of convergence over its neighborhood, since oscillations may also occur. When that convergence may be guaranteed will be treated in the next section.

4. Convergence theorems for standardized, recurrent fuzzy systems

We shall start off by taking up the case of standardized, recurrent fuzzy systems, which have no input parameters, and will, in the following, thus be termed “autonomous, standardized, recurrent fuzzy systems,” whose transition functions will be designated by $\mathbf{f}(\mathbf{x})$. Statements regarding such systems are directly applicable to standardized, recurrent fuzzy systems provided that their input vector is fixed and coincides with a core position vector. Such systems will be governed by a single, fixed, set of rules, as in the case of autonomous, standardized, recurrent fuzzy systems, instead of several, simultaneously active, sets of rules.

Convergence of a recurrent fuzzy system is defined as convergence of the sequence of state vectors, $\mathbf{x}(k)$, from Eq. (5). In the case of autonomous, recurrent fuzzy systems, the state vector, $\mathbf{x}(k)$, is obtained from the starting vector, $\mathbf{x}(0)$, by applying the transition function, $\mathbf{f}$, k-times, which will be designated using the abbreviated notation $\mathbf{f}^k$, i.e., $\mathbf{x}(k) = \mathbf{f}^k(\mathbf{x}(0))$. Analogously, in the case of nonautonomous, recurrent fuzzy systems, $\mathbf{f}^k(\mathbf{x}, \mathbf{u})$ will be used to designate the k-times application of the transition function, $\mathbf{f}$, for a fixed input vector, $\mathbf{u}$.

A necessary condition for a recurrent fuzzy system to converge for any, arbitrary, starting vector, $\mathbf{x}$, is that no true cycles, i.e., sequences of points that periodically reassume the same values and contain no fixed points, occur. In particular, there can be no true cycles in their rule base. It follows that the linguistic automaton will map every one of its linguistic state vectors onto an end node at some point in time, i.e., will map them onto a linguistic vector, which, in turn, will be mapped onto itself. A recurrent fuzzy system that complies with this condition is termed a “terminated, recurrent fuzzy system.” We then have that

Theorem 4. *Autonomous, terminated, rule-continuous, standardized, recurrent fuzzy systems converge for any initial state value.*

Proof. For an accumulation point, $\mathbf{x}$, of an arbitrary time series, $\mathbf{a}(k) = \mathbf{f}^k(\mathbf{a})$, of an autonomous, terminated, rule-continuous, standardized, recurrent fuzzy system, we have that

$$\liminf_{k\to\infty} \|\mathbf{f}^k(\mathbf{a}) - \mathbf{x}\| = 0. \tag{27}$$

According to Lemma 6 in Appendix A, the vector $\mathbf{x}$ is a fixed point, and we therefore obtain

$$\liminf_{k\to\infty} \|\mathbf{f}^k(\mathbf{a}) - \mathbf{f}^k(\mathbf{x})\| = \liminf_{k\to\infty} \|\mathbf{f}^k(\mathbf{a}) - \mathbf{x}\| = 0. \tag{28}$$

According to Theorem 2, the function $\mathbf{f}$ is nonexpansive, and we therefore obtain

$$\lim_{k\to\infty} \|\mathbf{f}^k(\mathbf{a}) - \mathbf{f}^k(\mathbf{x})\| = \liminf_{k\to\infty} \|\mathbf{f}^k(\mathbf{a}) - \mathbf{f}^k(\mathbf{x})\| = 0. \tag{29}$$

According to Lemma 6 in the appendix, the vector $\mathbf{x}$ is a fixed point, and therefore we finally obtain

$$\lim_{k\to\infty} \|\mathbf{f}^k(\mathbf{a}) - \mathbf{x}\| = \lim_{k\to\infty} \|\mathbf{f}^k(\mathbf{a}) - \mathbf{f}^k(\mathbf{x})\| = 0 \tag{30}$$

and an arbitrarily chosen time series, $\mathbf{f}^k(\mathbf{a})$, will converge to $\mathbf{x}$. □

Each linguistic state vector of the associated linguistic automaton of an autonomous, terminated, rule-continuous, standardized, recurrent fuzzy system will arrive at an end node, i.e., it will converge. According to Theorem 4, that convergence will transfer itself to the state vectors, $\mathbf{x}$, of the recurrent fuzzy system. Conclusions regarding the dynamic behavior of the recurrent fuzzy system may thus be drawn, based on the behavior of the associated linguistic automaton, using the latter’s state graph.

The conditions of Theorem 4 may not be readily relaxed. As explained above, a convergent, recurrent fuzzy system is, necessarily, terminated. If only the rule-continuity condition is violated, then nonconvergent, recurrent fuzzy systems, like that of the example depicted in Fig. 8, which will be discussed in another conjunction later, exist. Furthermore, there are (nonautonomous) terminated, rule-continuous, standardized, recurrent fuzzy systems that are nonconvergent. However, convergence can, nevertheless, be guaranteed for a major class of nonautonomous, terminated, rule-continuous, standardized, recurrent fuzzy systems, namely, in the case of one-dimensional systems, i.e., systems whose state vectors have a single component only, as the following theorem shows:

Theorem 5. *One-dimensional, terminated, rule-continuous, standardized, recurrent fuzzy systems converge for any initial state value and any given fixed input vector.*

Proof. Assume that a given one-dimensional, terminated, rule-continuous, standardized, recurrent fuzzy system does not converge for a given fixed input vector, $\mathbf{u}$. According to [11], its transition function, f, has a 2-periodic point, i.e., there exists a point, $a \in X$, with $f(a,\mathbf{u}) \neq a$ and $f^2(a,\mathbf{u}) = a$. According to Theorem 1, the transition function, f, has a fixed point, b, i.e., $f(b,\mathbf{u}) = b$. As the function f is nonexpansive according to Theorem 2, we have

$$\|f^2(a,\mathbf{u}) - f^2(b,\mathbf{u})\| \leqslant \|f(a,\mathbf{u}) - f(b,\mathbf{u})\| \leqslant \|a-b\|. \tag{31}$$

Using the properties of a and b, Eq. (31) can be rewritten as

$$\|a-b\| \leqslant \|f(a,\mathbf{u}) - b\| \leqslant \|a-b\| \tag{32}$$

and we obtain

$$\|f(a,\mathbf{u}) - b\| = \|a-b\|. \tag{33}$$

Since the transition function, f, is one dimensional, it follows that

$$f(a,\mathbf{u}) - b = \pm(a-b). \tag{34}$$

Since a is not a fixed point, we get that

$$f(a,\mathbf{u}) - b = b - a. \tag{35}$$

Without loss of generality, we may assume that $f(a,\mathbf{u}) < b < a$, since the points a and $f(a,\mathbf{u})$ are points of the same 2-periodic cycle. Eq. (35) is also valid for all values, x, in $[f(a,\mathbf{u}), a]$, as can be seen as follows: Employing the triangular inequality and the nonexpansion property of the function f we have for all $x \in [b,a]$ that

$$\begin{aligned}\|a-b\| = \|a-x\| + \|x-b\| &\geqslant \|f(a,\mathbf{u}) - f(x,\mathbf{u})\| + \|f(x,\mathbf{u}) - b\| \\ &\geqslant \|f^2(a,\mathbf{u}) - f^2(x,\mathbf{u})\| + \|f^2(x,\mathbf{u}) - b\| \\ &\geqslant \|f^2(a,\mathbf{u}) - b\|.\end{aligned} \tag{36}$$

Since a is a 2-periodic point, the first and the last term of the chain inequality (36) are equal. Therefore, all terms in the chain inequality (36) are equal as well and we have

$$\|a-x\| + \|x-b\| = \|f(a,\mathbf{u}) - f(x,\mathbf{u})\| + \|f(x,\mathbf{u}) - b\|. \tag{37}$$

According to Theorem 2, the inequalities $\|a-x\| \leqslant \|f(a,\mathbf{u})-f(x,\mathbf{u})\|$ and $\|x-b\| \leqslant \|f(x,\mathbf{u})-b\|$ are valid, thus we have $\|x-b\| = \|f(x,\mathbf{u})-b\|$ in order to satisfy Eq. (37) and therefore

$$f(x,\mathbf{u})-b = \pm(x-b). \tag{38}$$

Since f is a continuous function, it follows from Eq. (35) that

$$f(x,\mathbf{u}) = -x+2b \tag{39}$$

for all $x \in [b,a]$ and analogously for all $x \in [f(a,\mathbf{u}),b]$ and therefore for all $x \in [f(a,\mathbf{u}),a]$.

The fixed point, b, lies between the two core positions s_r^x and s_{r+1}^x, i.e., $s_r^x \leqslant b < s_{r+1}^x$. In the interval $[s_r^x, s_{r+1}^x]$ only the membership functions $\mu_r^x(x)$ and $\mu_{r+1}^x(x)$ differ from zero. Then together with Eq. (48) of Lemma 2 in Appendix A, it follows for all $x \in [b, s_{r+1}^x] \cap [b,a]$ that

$$f(x,\mathbf{u}) = \sum_j f(s_j^x,\mathbf{u})\mu_j^x(x) = f(s_r^x,\mathbf{u})\mu_r^x(x) + f(s_{r+1}^x,\mathbf{u})\mu_{r+1}^x(x). \tag{40}$$

Since the membership functions are normalized, we have $\mu_{r+1}^x(x) = 1-\mu_r^x(x)$ according to Eq. (4). Furthermore, the triangular membership function is given by $\mu_r^x(x) = -(x-s_{r+1}^x)/\Delta s$ where $\Delta s = s_{r+1}^x - s_r^x$. Thus, we can rewrite Eq. (40) by

$$\begin{aligned} f(x,\mathbf{u}) &= (f(s_r^x,\mathbf{u})-f(s_{r+1}^x,\mathbf{u}))\mu_r^x(x) + f(s_{r+1}^x,\mathbf{u}) \\ &= -\frac{1}{\Delta s}(f(s_r^x,\mathbf{u})-f(s_{r+1}^x,\mathbf{u}))x + \frac{1}{\Delta s}(f(s_r^x,\mathbf{u})-f(s_{r+1}^x,\mathbf{u}))s_{r+1}^x + f(s_{r+1}^x,\mathbf{u}). \end{aligned} \tag{41}$$

Comparing the coefficients of x in Eqs. (39) and (41), it follows that

$$f(s_r^x,\mathbf{u}) - f(s_{r+1}^x,\mathbf{u}) = \Delta s. \tag{42}$$

Due to the rule-continuity condition, we have $s_{w(r,\mathbf{q})}^x - s_{w(r+1,\mathbf{q})}^x \leqslant \Delta s$. If the strict inequality $s_{w(r,\mathbf{q})}^x - s_{w(r+1,\mathbf{q})}^x < \Delta s$ was satisfied for an index vector $\mathbf{q}$ with $\mu_{q_p}^{u_p}(u_p) \neq 0$, we would be able to derive $f(s_r^x,\mathbf{u}) - f(s_{r+1}^x,\mathbf{u}) = \sum_{\mathbf{q}}(s_{w(r,\mathbf{q})}^x - s_{w(r+1,\mathbf{q})}^x)\mu_{q_p}^{u_p}(u_p) < \Delta s$ from Eq. (48) of Lemma 2 in Appendix A in contradiction to Eq. (42). Therefore we have for all $\mathbf{q}$ with $\mu_{q_p}^{u_p}(u_p) \neq 0$:

$$s_{w(r,\mathbf{q})}^x - s_{w(r+1,\mathbf{q})}^x = \Delta s. \tag{43}$$

Two cases have to be considered:

Case 1: If $b = s_r^x$, then an index vector, $\mathbf{q}$, exists satisfying $\prod_p \mu_{q_p}^{u_p}(u_p) \neq 0$ and $s_{w(r,\mathbf{q})}^x = f(b,\mathbf{u}) = b = s_r^x$, due to the rule-continuity condition. According to Eq. (43), we get $s_{w(r+1,\mathbf{q})}^x = s_{w(r,\mathbf{q})}^x - \Delta s = s_r^x - \Delta s = s_{r-1}^x$ and analogously $s_{w(r-1,\mathbf{q})}^x = s_{r+1}^x$, and therefore

$$s_{w(r+1,\mathbf{q})}^x = s_{r-1}^x \quad \text{and} \quad s_{w(r-1,\mathbf{q})}^x = s_{r+1}^x. \tag{44}$$

Thus the recurrent fuzzy system has a real cycle for the linguistic input vector $\mathbf{L}_{\mathbf{q}}^{\mathbf{u}}$ characterized by the index vector $\mathbf{q}$. Therefore the rule base is not terminated in contradiction to the premise of the theorem.

Case 2: Assume $b > s_r^x$. The transition function, $\mathbf{f}$, is monotonic increasing or decreasing in the variable x between two core positions and due to Eq. (42) monotonic decreasing between s_r^x and s_{r+1}^x. Since b is a fixed point in the open interval (s_r^x, s_{r+1}^x), it follows that $f(s_r^x,\mathbf{u}) > b > f(s_{r+1}^x,\mathbf{u})$. Due to the rule-continuity condition and Eq. (43), an index vector, $\mathbf{q}$, exists satisfying $\prod_p \mu_{q_p}^{u_p}(u_p) \neq 0$ and $s^x_{w(r,\mathbf{q})} > b > s^x_{w(r+1,\mathbf{q})}$. Since there is no further core position between $s^x_{w(r,\mathbf{q})}$ and $s^x_{w(r+1,\mathbf{q})}$ according to Eq. (43) and $s^x_r < b < s^x_{r+1}$, it follows that

$$s^x_{w(r+1,\mathbf{q})} = s^x_r \quad \text{and} \quad s^x_{w(r,\mathbf{q})} = s^x_{r+1}. \tag{45}$$

Thus the recurrent fuzzy system is not terminated in this case, in contradiction to the premise of the theorem.

In both cases, we obtain a contradiction. Thus the assumption is wrong, and the recurrent fuzzy system converges. □

By using the convergence theorems, it is possible to investigate whether a given system converges for all initial state values to a final value. Where those final values may be found will be discussed in the next section.

5. Locating equilibrium points

As described under Lemma 1 appearing in Section 3, some equilibrium points, i.e., fixed points, may be immediately read off from the rule base. In the following, that statement will be extended to points other than core position vectors, which will require correlating arbitrary points, $\mathbf{x}$, to core position vectors, which may be accomplished using the definition of the elementary hypersquare of $\mathbf{x}$ [4], which, in the case of autonomous, standardized, recurrent fuzzy systems, may also be described as follows:

Consider the set of all index vectors, $\mathbf{j}$, for which $\prod_i \mu_{j_i}^{x_i}(x_i) \neq 0$. Those core position vectors, $\mathbf{s}^{\mathbf{x}}_{\mathbf{j}}$, that have these index vectors, $\mathbf{j}$, form the corners of a hypersquare in the state space, X, that is termed an "elementary hypersquare of $\mathbf{x}$." We then have that:

Theorem 6. *Any equilibrium point in an autonomous, terminated, rule-continuous, standardized, recurrent fuzzy systems is located in an elementary hypersquare, whose vertices are equilibrium points themselves, i.e., core position vectors of end nodes. Furthermore, such an elementary hypersquare consists of equilibrium points.*

Proof. Any equilibrium point, $\mathbf{x}$, is a fixed point of the transition function. According to Lemma 6 in the appendix, the elementary hypersquare of $\mathbf{x}$ consists of equilibrium points. In particular, the vertices of the elementary hypersquare of $\mathbf{x}$ are fixed points, as well, and therefore also core position vectors of end nodes. □

Theorem 6 yields a result that may be expected for recurrent fuzzy systems: equilibrium points occur exclusively in those subregions of their state space where the core positions of end nodes occur. Fig. 6 depicts the state graphs of, and equilibrium points occurring in the state spaces of several simple, terminated, rule-continuous, standardized, recurrent fuzzy systems. The equilibrium points appearing in Figs. 6(a) and (d) have already been described by Lemma 1, since they coincide with core position vectors of end nodes. Figs. 6(b) and (e) show that even high-order, multidimensional, elementary hypersquares may consist of large numbers of equilibrium points if their corners are core

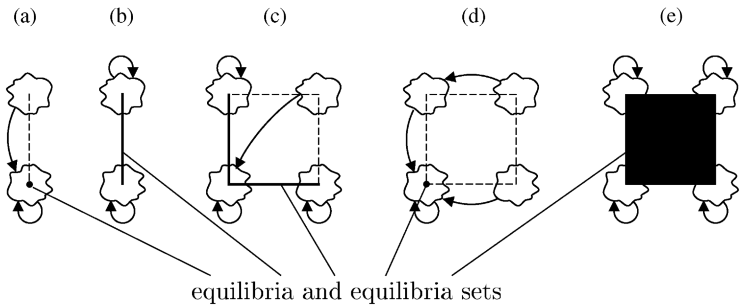

Fig. 6. Examples for equilibria sets given by various rule bases of autonomous, terminated, rule-continuous, standardized, recurrent fuzzy systems according to Theorem 6. The dashed lines indicate the borders of hypersquares.

position vectors of end nodes. It may be seen from Fig. 6(c) that a rule base may also have several elementary hypersquares consisting of equilibrium points.

For unterminated systems, a similar criterion applies to their equilibrium points. Since true cycles may then occur in their rule base, the corners of the elementary hypersquare of an equilibrium point are not necessarily fixed points. Nevertheless, the equilibrium points of such systems may be readily located, as the following theorem shows:

Theorem 7. *Any equilibrium point in an autonomous, rule-continuous, standardized, recurrent fuzzy system is located in an elementary hypersquare, whose vertices are permutated by the transition function.*

Proof. Assume that $\mathbf{x}$ is an equilibrium point in an autonomous, rule-continuous, standardized, recurrent fuzzy system, and H is its elementary hypersquare. Then a vertex, $\hat{\mathbf{s}}$, of H exists, that has the smallest distance to the point $\mathbf{x}$ of all core position vectors in X. Since the transition function, $\mathbf{f}$, is nonexpansive according to Theorem 2, we have that $\|\mathbf{f}(\hat{\mathbf{s}}) - \mathbf{f}(\mathbf{x})\| \leqslant \|\hat{\mathbf{s}} - \mathbf{x}\|$. Since the image point $\mathbf{f}(\hat{\mathbf{s}})$ is a core position vector itself and $\mathbf{x}$ is a fixed point, the two image points, $\mathbf{f}(\hat{\mathbf{s}})$ and $\mathbf{f}(\mathbf{x})$, have the same distance as $\hat{\mathbf{s}}$ and $\mathbf{x}$. Furthermore the elementary hypersquare of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ are identical, since $\mathbf{x}$ is a fixed point. Therefore, Lemmas 4 and 5 in Appendix A are applicable in their extended version.

According to Lemma 5, the set, S, of vertices of H is mapped onto itself. Furthermore no vertex of these but $\hat{\mathbf{s}}$ is mapped onto $\mathbf{f}(\hat{\mathbf{s}})$ due to Lemma 4. Now we regard the set of all vertices of H excluding $\mathbf{e}_1 = \hat{\mathbf{s}}$. Again we choose a vertex, $\mathbf{e}_2$, from this set that has the smallest distance to $\mathbf{x}$. Analogously to the case of $\mathbf{e}_1$, we conclude that no other vertex but $\mathbf{e}_2$ is mapped onto $\mathbf{f}(\mathbf{e}_2)$. By induction on all points of S, we conclude that there are no two points of S that are mapped onto the same vertex in S by the transition function. As the set, S, of vertices of H is finite, the elements are permutated by the transition function. □

In order to illustrate this theorem, Fig. 7 depicts all possible one- and two-dimensional elementary hypersquares of equilibrium points in autonomous, rule-continuous, standardized, recurrent fuzzy systems. The applicable rules are given by their state graphs. Their sets of equilibrium points consist of points, lines, and/or surfaces. In the case of unterminated systems, their sets of equilibrium points

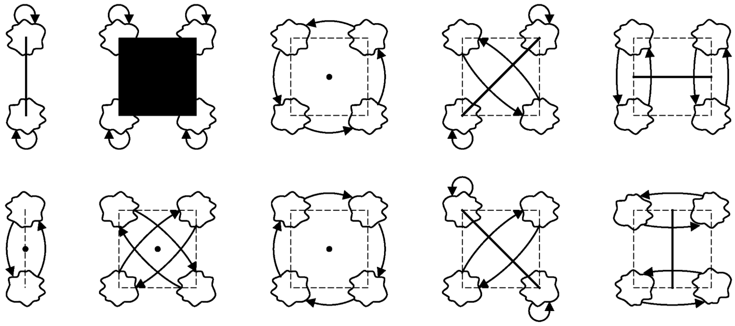

Fig. 7. Schematic of all possible one- and two-dimensional elementary hypersquares, in which autonomous, rule-continuous, standardized, recurrent fuzzy systems have equilibrium points according to Theorem 7.

are invariably subsets of hypersquares. Moreover, the centroid of every such elementary hypersquare is also an equilibrium point, as may readily be proven.

Once such an equilibrium point has been found, its stability is the next matter of interest, a matter that will be investigated in the next section.

6. Asymptotic stability and exclusion of chaos

We have thus far considered the matters of whether equilibrium points exist and how they may be located. Theorem 3 also yields a statement on the stabilities of the equilibrium points of rule-continuous, standardized, recurrent fuzzy systems: they are all stable in the sense of Liapunov. The following theorem, Theorem 8, describes the effect of the existence of an asymptotically stable equilibrium point on the dynamics of the overall system.

Theorem 8. *An asymptotically stable equilibrium point in a rule-continuous, standardized, recurrent fuzzy system is also globally asymptotically stable.*

Proof. Assume $\mathbf{b}$ is an asymptotically stable equilibrium point for a given fixed input vector, $\mathbf{u}$. Then for a suitably small, fixed, $\varepsilon > 0$ an ε-neighborhood $U_\varepsilon(\mathbf{b}) = \{\mathbf{x} \in X \mid \|\mathbf{x} - \mathbf{b}\| < \varepsilon\}$ exists such that for all points $\mathbf{x} \in U_\varepsilon(\mathbf{b})$ the series $\mathbf{f}^k(\mathbf{x}, \mathbf{u})$ converges to $\mathbf{b}$.

Now we choose a state vector $\mathbf{x} \in X$ arbitrarily. The vector $\mathbf{x}$ is contained in a neighborhood $U_{v(\varepsilon/2)}(\mathbf{b}) = \{\mathbf{x} \in X \mid \|\mathbf{x} - \mathbf{b}\| < v \cdot \varepsilon/2\}$, with a minimal parameter $v \in \mathbb{N}$. In the following, we will show by induction that the vectors $\mathbf{f}^k(\mathbf{x}, \mathbf{u})$ converge to $\mathbf{b}$ as well:

In the cases of $v = 1$ and 2 the statement is true by definition. Now we assume that the statement holds up to a certain value of $v \in \mathbb{N}$. In the following induction step, we show that in this case the statement is also true for $v + 1$.

Choose a vector $\mathbf{x} \in U_{(v+1)(\varepsilon/2)}(\mathbf{b})$ arbitrarily. Find a vector $\mathbf{a} \in U_{v(\varepsilon/2)}(\mathbf{b})$ satisfying $\|\mathbf{a} - \mathbf{x}\| \leqslant \varepsilon/2$. Then it follows from the induction hypothesis that $\lim_{k \to \infty} \mathbf{f}^k(\mathbf{a}, \mathbf{u}) = \mathbf{b}$, and therefore that all vectors $\mathbf{f}^k(\mathbf{a}, \mathbf{u})$ lie within $U_{\varepsilon/2}(\mathbf{b})$ for all $k > k_0$ and a certain integer k_0. As the transition function, $\mathbf{f}$, is nonexpansive according to Theorem 2, $\|\mathbf{f}^k(\mathbf{a}, \mathbf{u}) - \mathbf{f}^k(\mathbf{x}, \mathbf{u})\| \leqslant \varepsilon/2$ holds for all k. Thus the vectors $\mathbf{f}^k(\mathbf{x},\mathbf{u})$ lie in $U_\varepsilon(\mathbf{b})$ also for all $k > k_0$ and therefore converge to $\mathbf{b}$, as $\mathbf{b}$ is asymptotically stable. □

For autonomous recurrent fuzzy systems and for those, whose input vector, **u**, coincides with a core position, we can easily find asymptotical stable equilibria in rule-continuous, standardized, recurrent fuzzy systems as can be seen in the following theorem:

Theorem 9. *In autonomous, rule-continuous, standardized, recurrent fuzzy systems each asymptotically stable equilibrium point coincides with a core position vector and the following statements are equivalent*:

(1) *An asymptotically stable equilibrium point exists.*
(2) *A (unique) globally asymptotically stable equilibrium point exists.*
(3) *A unique equilibrium point exists, and the recurrent fuzzy system converges.*
(4) *A unique end node exists, and the recurrent fuzzy system is terminated.*
(5) *The recurrent fuzzy system is terminated, and an end node exists, that has no end nodes adjacent to it.*
(6) *All linguistic vectors adjacent to an (the unique) end node converge to this end node.*

Proof. The equivalence of Statements (1)–(6) is shown by the following chain of implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (1). Then we can derive any of these statements from any other of these statements and therefore all statements are equivalent.

According to Theorem 8, Statement (1) implies Statement (2). Furthermore, the implication (2) ⇒ (3) is trivial.

Assume Statement (3) holds. The recurrent fuzzy system has to be terminated, since it is convergent. Thus an end node exists. The core position vector of an end node is a equilibrium point, as well. Since the equilibrium point is unique, only one end node exists. Therefore Statement (4) holds.

Assume Statement (4) holds. Since there is only one end node, there cannot be another end node adjacent to it. Therefore Statement (5) holds.

Assume Statement (5) holds. All linguistic vectors adjacent to an end node are mapped either onto this end node or again onto a linguistic vectors adjacent to this end node due to the rule-continuity condition. Furthermore, they have to converge to an end node, since the recurrent fuzzy system is terminated, and thus have to converge to this particular end node, since there is no end node adjacent to it. Therefore, Statement (6) holds.

Assume Statement (6) holds. According to Theorem 6, there is no further equilibrium point in the ε-neighborhood of the core position vector of an end node with $\varepsilon \leqslant 1$, as otherwise the vertices of the elementary hypersquare of such an equilibrium point would be equilibrium points themselves, and therefore there would be end nodes adjacent to the former end node in contradiction to Statement (6). Vectors in the ε-neighborhood of the core position vector of the end node cannot leave this neighborhood due to the nonexpansive transition function **f**, but converge to an equilibrium point according to Theorem 4. Therefore they converge to the only equilibrium point in the ε-neighborhood and this equilibrium point is asymptotically stable. Therefore Statement (1) holds.

The chain of implications is proven and Statements (1)–(6) are equivalent. If an asymptotically stable equilibrium point exists, cf. Statement (1), then it is unique according to Statement (2) and it is a core position vector of an end node, as well, in order to satisfy Statement (4). Therefore any asymptotically stable equilibrium point is a core position vector. □

Theorem 9 may be used to determine whether such systems have a globally asymptotically stable equilibrium point and where it will be found in a very simple manner, based on their rule base or state graph alone. If a globally asymptotically stable equilibrium point exists, then it has to be the core position vector of an end node. If there is an end node, then all that is needed is to employ Statement (6) of Theorem 9, i.e., to check whether all linguistic vectors adjacent to the end node converge to this end node. It then follows, firstly, that the entire rule base involved is terminated according to Statement (4), and, secondly, that the recurrent fuzzy system converges according to Statement (3) and has a globally asymptotically stable equilibrium point according to Statement (2).

Rule-continuous, standardized, recurrent fuzzy systems cannot only be simply investigated to determine whether they have asymptotically stable equilibrium points, but have another nice property:

Theorem 10. *Rule-continuous, standardized, recurrent fuzzy systems are not chaotic.*

Proof. There are various definitions of chaos in discrete-time dynamical systems. The most well known are chaos in the sense of Li and Yorke [19,24], chaos in the sense of Devaney [7] and chaos in the sense of Block and Coppel [6].

From these different definitions of chaos one common property can be derived: If a function is chaotic, a finite distance $\delta > 0$ exists such that initial state values, $\mathbf{a}$ and $\mathbf{b}$, can be found arbitrarily close to each other, whose images will have a distance that exceeds the distance δ after a number of iterations. The distances are measured by a norm $\|..\|_s$ of the difference vectors of $\mathbf{f}^k(\mathbf{a},\mathbf{u})$ and $\mathbf{f}^k(\mathbf{b},\mathbf{u})$. As norms are equivalent in finite dimensional vector spaces [17], the norm $\|..\|_s$ and the canonical vector norm $\|..\|$ satisfy $m\|\mathbf{x}\|_s < \|\mathbf{x}\| < M\|\mathbf{x}\|_s$ with certain constant values $0 < m < M$ for arbitrary vectors $\mathbf{x}$.

Assume that a rule-continuous, standardized, recurrent fuzzy system is chaotic. Then a value $\delta > 0$ and initial state values, $\mathbf{a}$ and $\mathbf{b}$, with an initial distance $\|\mathbf{a}-\mathbf{b}\|_s < \delta m/M$ exists, whose images have a distance that would exceed δ after k iterations. Then the following chain inequality would hold:

$$\|\mathbf{a}-\mathbf{b}\| < M\|\mathbf{a}-\mathbf{b}\|_s < m\delta < m\|\mathbf{f}^k(\mathbf{a},\mathbf{u})-\mathbf{f}^k(\mathbf{b},\mathbf{u})\|_s < \|\mathbf{f}^k(\mathbf{a},\mathbf{u})-\mathbf{f}^k(\mathbf{b},\mathbf{u})\|. \tag{46}$$

But the resulting inequality $\|\mathbf{a}-\mathbf{b}\| < \|\mathbf{f}^k(\mathbf{a},\mathbf{u})-\mathbf{f}^k(\mathbf{b},\mathbf{u})\|$ contradicts the chain inequality $\|\mathbf{a}-\mathbf{b}\| \geqslant \|\mathbf{f}(\mathbf{a},\mathbf{u})-\mathbf{f}(\mathbf{b},\mathbf{u})\| \geqslant \cdots \geqslant \|\mathbf{f}^k(\mathbf{a},\mathbf{u})-\mathbf{f}^k(\mathbf{b},\mathbf{u})\|$, which is obtained by employing Theorem 2 repeatedly. □

If the rule-continuity property is violated at a single location only, then chaotic behavior may well occur. Illustrating such a chaotic behavior, Fig. 8 depicts the state graph, (a), and the transition function, (b), of a chaotic, standardized, recurrent fuzzy system and an example of a chaotic time series, (c), for a given starting point, $x(0)$, of this recurrent fuzzy system. The chaotic behavior of

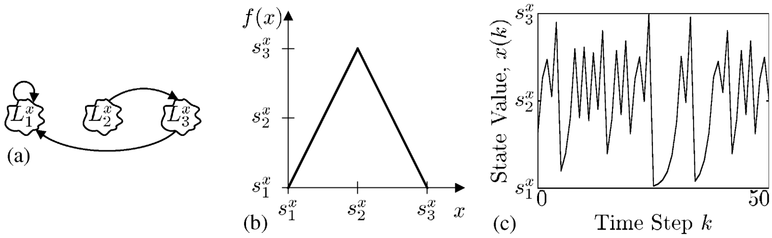

Fig. 8. State graph, (a) transition function, (b) and a time series for a given starting point, (c) of a chaotic, standardized, recurrent fuzzy system.

its transition function may be derived from a simple analysis of its state graph alone, as has been discussed in detail in [4]. Due to the lack of rule-continuity, its trajectories may drift apart for many choices of starting vectors, which may then lead to chaos.

7. Summary and conclusions

In this article, we have investigated recurrent fuzzy systems with regard to their equilibrium points. It has been shown how conclusions regarding the dynamic characteristics of recurrent fuzzy systems may be drawn from their rule base. To that end, a major subclass, that of standardized, recurrent fuzzy systems, has been considered. It has been found that rule-continuous, standardized, recurrent fuzzy systems are nonexpansive, i.e., that their trajectories cannot drift apart, and that all of their equilibrium points are thus stable, in the sense of Liapunov. Convergence theorems for standardized, recurrent fuzzy systems were subsequently presented, where the prerequisites of these theorems are verifiable, based on the rule bases or state graphs alone. It has also been shown how their equilibrium points may be found, based on their rule base or state graph. In concluding, we have described the effect that the existence of an asymptotically stable equilibrium point has on the dynamics of rule-continuous, standardized, recurrent fuzzy systems, and that chaos cannot occur in such systems.

Appendix A

This appendix contains Lemmata that are employed in some proofs of the theorems in this article. In Lemma 2, a number of equations are presented for the transition function.

Lemma 2. *The transition function*, $\mathbf{f}$, *of a recurrent fuzzy system is given by the values* $\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}, \mathbf{s}_\mathbf{q}^\mathbf{u})$ *at core position vectors, the membership functions*, $\mu_{j_i}^{x_i}(x_i)$ *and* $\mu_{q_p}^{u_p}(u_p)$, *and by the following equation*:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \sum_{\mathbf{j},\mathbf{q}} \mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}, \mathbf{s}_\mathbf{q}^\mathbf{u}) \prod_i \mu_{j_i}^{x_i}(x_i) \prod_p \mu_{q_p}^{u_p}(u_p). \tag{A.1}$$

The transition function, $\mathbf{f}$, *satisfies in addition the following equations*:

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \sum_{\mathbf{j}} \mathbf{f}(\mathbf{s}_{\mathbf{j}}^{\mathbf{x}},\mathbf{u}) \prod_i \mu_{j_i}^{x_i}(x_i), \tag{A.2}$$

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \sum_{\mathbf{q}} \mathbf{f}(\mathbf{x},\mathbf{s}_{\mathbf{q}}^{\mathbf{u}}) \prod_p \mu_{q_p}^{u_p}(u_p). \tag{A.3}$$

Proof. Eq. (A.1) is obtained, by inserting $\mathbf{f}(\mathbf{s}_{\mathbf{j}}^{\mathbf{x}},\mathbf{s}_{\mathbf{q}}^{\mathbf{u}}) = \mathbf{s}_{\mathbf{w}(\mathbf{j},\mathbf{q})}^{\mathbf{x}}$ from Eq. (6) into Eq. (5). Then Eq. (A.1) holds, and we get particularly for $\mathbf{x} = \mathbf{s}_{\mathbf{j}}^{\mathbf{x}}$:

$$\mathbf{f}(\mathbf{s}_{\mathbf{j}}^{\mathbf{x}},\mathbf{u}) = \sum_{\mathbf{q}} \mathbf{f}(\mathbf{s}_{\mathbf{j}}^{\mathbf{x}},\mathbf{s}_{\mathbf{q}}^{\mathbf{u}}) \prod_p \mu_{q_p}^{u_p}(u_p). \tag{A.4}$$

We obtain again Eq. (A.1) by inserting Eq. (A.4) into Eq. (A.2), which is an equivalent transformation of Eq. (A.2). Therefore Eq. (A.2) holds. Analogously we can derive Eq. (A.3). □

As discussed in the article, triangular membership functions stand out for a property shown in Lemma 3.

Lemma 3. *The following equations hold for recurrent fuzzy systems, if and only if triangular membership functions are chosen for the state vectors*:

$$\mathbf{x} = \sum_{\mathbf{j}} \mathbf{s}_{\mathbf{j}}^{\mathbf{x}} \prod_i \mu_{j_i}^{x_i}(x_i) = \sum_{\mathbf{j},\mathbf{q}} \mathbf{s}_{\mathbf{j}}^{\mathbf{x}} \prod_i \mu_{j_i}^{x_i}(x_i) \prod_p \mu_{q_p}^{u_p}(u_p). \tag{A.5}$$

Proof. For each index q_p the terms $\sum_{q_p} \mu_{q_p}^{u_p}(u_p)$ can be factored out from the left-hand side of Eq. (A.5), since the product $\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} \prod_i \mu_{j_i}^{x_i}(x_i) \prod_p \mu_{q_p}^{u_p}(u_p)$ contains one and only one factor, i.e., $\mu_{q_p}^{u_p}(u_p)$, that depends on the index q_p and since this factor does not depend on any other index. Due to the normalization condition from Eq. (4), the term $\sum_{q_p} \mu_{q_p}^{u_p}(u_p)$ is unity for each indices q_p and therefore may be left out. Thus, we have in general that

$$\sum_{\mathbf{j}} \mathbf{s}_{\mathbf{j}}^{\mathbf{x}} \prod_i \mu_{j_i}^{x_i}(x_i) = \sum_{\mathbf{j},\mathbf{q}} \mathbf{s}_{\mathbf{j}}^{\mathbf{x}} \prod_i \mu_{j_i}^{x_i}(x_i) \prod_p \mu_{q_p}^{u_p}(u_p). \tag{A.6}$$

Now assume that Eq. (A.5) holds. Choose an arbitrary component, x_h, of $\mathbf{x}$. Following the same arguments for the indices $j_i \neq j_h$ as we did above for the indices q_p, we can simplify Eq. (A.5) and obtain:

$$x_h = \sum_{\mathbf{j}} s_{j_h}^{x_h} \prod_i \mu_{j_i}^{x_i}(x_i) = \sum_{j_h} s_{j_h}^{x_h} \mu_{j_h}^{x_h}(x_h). \tag{A.7}$$

The point x_h lies between two core positions, $s_r^{x_h}$ and $s_{r+1}^{x_h}$. Thus the membership functions $\mu_{j_h}^{x_h}(x_h)$ are positive for no more than the two indices $j_h = r$ and $r+1$ and we have according to Eq. (4) that

$$\mu_{r+1}^{x_h}(x_h) = 1 - \mu_r^{x_h}(x_h). \tag{A.8}$$

Employing this, Eq. (A.7) can be written as:

$$x_h = \sum_{j_h} s_{j_h}^{x_h} \mu_{j_h}^{x_h}(x_h) = s_r^{x_h} \mu_r^{x_h}(x_h) + s_{r+1}^{x_h} \mu_{r+1}^{x_h}(x_h) = (s_r^{x_h} - s_{r+1}^{x_h}) \mu_r^{x_h}(x_h) + s_{r+1}^{x_h}. \tag{A.9}$$

Solving this equation for $\mu_r^{x_h}(x_h)$, the membership function satisfies the following equation of the straight line in the interval $[s_r^{x_h}, s_{r+1}^{x_h}]$:

$$\mu_r^{x_h}(x_h) = \frac{s_{r+1}^{x_h} - x_h}{s_{r+1}^{x_h} - s_r^{x_h}}. \tag{A.10}$$

This equation of the straight line describes one flank of a triangular membership function and thus Eq. (A.5) is only valid for triangular membership functions.

On the other side, if we use triangular membership functions, Eq. (A.10) is satisfied and can be transformed into Eq. (A.7), which is equivalent to Eq. (A.5), as has been shown above. □

The left-hand side of Eq. (A.5) can be interpreted as the transition function, **f**, of the 'verbal identical mapping', as can be verified by inserting $\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}, \mathbf{s}_\mathbf{q}^\mathbf{u}) = \mathbf{s}_\mathbf{j}^\mathbf{x}$ into Eq. (3). Lemma 3 shows that the verbal identical mapping results in the identical mapping, i.e., $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{x}$, only if triangular membership functions are employed.

In the following Lemmata 4–6 we use the term "elementary hypersquare of a vector **x**." For its definition we consider the set of all index vectors, **j**, for which $\prod_i \mu_{j_i}^{x_i}(x_i) \neq 0$. Those core position vectors, $\mathbf{s}_\mathbf{j}^\mathbf{x}$, that have these index vectors, **j**, form the vertices of a hypersquare in the state space, X, that is termed an "elementary hypersquare of **x**."

Lemma 4. *If a state vector*, **x**, *and a core position vector*, $\hat{\mathbf{s}} \in X$, *in an autonomous*, *rule-continuous*, *standardized*, *recurrent fuzzy system satisfy the equation* $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{x} - \hat{\mathbf{s}}\|$, *then*

$$\|\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}) - \mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{s}_\mathbf{j}^\mathbf{x} - \hat{\mathbf{s}}\| \tag{A.11}$$

holds for all core position vectors $\mathbf{s}_\mathbf{j}^\mathbf{x}$ *that are vertices of the elementary hypersquare of* **x**.

Proof. The set $A = \{\mathbf{j} | \prod_i \mu_{j_i}^{x_i}(x_i) \neq 0\}$ consists of the indices of all vertices of the elementary hypersquare of **x**. From Lemma 2 we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\hat{\mathbf{s}})\| = \left\| \sum_{\mathbf{j} \in A} \mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}) \prod_i \mu_{j_i}^{x_i}(x_i) - \mathbf{f}(\hat{\mathbf{s}}) \right\|. \tag{A.12}$$

Since the membership functions are normalized, i.e., $\sum_{\mathbf{j} \in A} \prod_i \mu_{j_i}^{x_i}(x_i) = \prod_i \sum_{j_i \in A} \mu_{j_i}^{x_i}(x_i) = 1$, it follows that

$$\left\| \sum_{\mathbf{j} \in A} \mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}) \prod_i \mu_{j_i}^{x_i}(x_i) - \mathbf{f}(\hat{\mathbf{s}}) \right\| = \left\| \sum_{\mathbf{j} \in A} (\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}) - \mathbf{f}(\hat{\mathbf{s}})) \prod_i \mu_{j_i}^{x_i}(x_i) \right\|$$

$$\leqslant \sum_{\mathbf{j} \in A} \|\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}) - \mathbf{f}(\hat{\mathbf{s}})\| \prod_i \mu_{j_i}^{x_i}(x_i). \tag{A.13}$$

Since $\mathbf{f}$ is nonexpansive according to Theorem 2, it follows that $\|\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| \leqslant \|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\|$ and thus we have

$$\sum_{\mathbf{j}\in A} \|\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| \prod_i \mu_{j_i}^{x_i}(x_i) \leqslant \sum_{\mathbf{j}\in A} \|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\| \prod_i \mu_{j_i}^{x_i}(x_i)$$
$$= \sum_{\mathbf{j}\in A}\sum_l \frac{1}{\Delta s_l}|s_{j_l}^{x_l}-\hat{s}_l| \prod_i \mu_{j_i}^{x_i}(x_i). \tag{A.14}$$

Since $1/\Delta s_l(s_{j_l}^{x_l}-\hat{s}_l)$ is an integer and the values $1/\Delta s_l(s_{j_l}^{x_l}-\hat{s}_l)$ differ by 1 at most for all $\mathbf{j}\in A$, it follows that for each index l we either have $1/\Delta s_l(s_{j_l}^{x_l}-\hat{s}_l)\leqslant 0$ for all $\mathbf{j}\in A$ or $1/\Delta s_l(s_{j_l}^{x_l}-\hat{s}_l)\geqslant 0$ for all $\mathbf{j}\in A$ and therefore

$$\sum_{\mathbf{j}\in A}\sum_l \frac{1}{\Delta s_l}|s_{j_l}^{x_l}-\hat{s}_l| \prod_i \mu_{j_i}^{x_i}(x_i) = \sum_l \frac{1}{\Delta s_l}\left|\sum_{\mathbf{j}\in A}(s_{j_l}^{x_l}-\hat{s}_l)\prod_i \mu_{j_i}^{x_i}(x_i)\right|. \tag{A.15}$$

Using Lemma 3, it follows that

$$\sum_l \frac{1}{\Delta s_l}\left|\sum_{\mathbf{j}\in A}(s_{j_l}^{x_l}-\hat{s}_l)\prod_i \mu_{j_i}^{x_i}(x_i)\right| = \sum_l \frac{1}{\Delta s_l}\left|\sum_{\mathbf{j}\in A} s_{j_l}^{x_l}\prod_i \mu_{j_i}^{x_i}(x_i)-\hat{s}_l\right|$$
$$= \left\|\sum_{\mathbf{j}\in A} \mathbf{s}_\mathbf{j}^\mathbf{x}\prod_i \mu_{j_i}^{x_i}(x_i)-\hat{\mathbf{s}}\right\| = \|\mathbf{x}-\hat{\mathbf{s}}\|. \tag{A.16}$$

Combining Eqs. (A.12)–(A.16), we get $\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| \leqslant \|\mathbf{x}-\hat{\mathbf{s}}\|$ and from the premise of the lemma we have $\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{x}-\hat{\mathbf{s}}\|$. Thus all terms in Eqs. (A.12)–(A.16) are equal and it follows that

$$\sum_{\mathbf{j}\in A} \|\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| \prod_i \mu_{j_i}^{x_i}(x_i) = \sum_{\mathbf{j}\in A} \|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\| \prod_i \mu_{j_i}^{x_i}(x_i). \tag{A.17}$$

Since $\mathbf{f}$ is nonexpansive, we have $\|\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| \leqslant \|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\|$. Since we have $\prod_i \mu_{j_i}^{x_i}(x_i) \neq 0$ for all $\mathbf{j}\in A$,

$$\|\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\| \tag{A.18}$$

holds for all $\mathbf{j}\in A$ in order to satisfy Eq. (A.17). □

Lemma 5. *If a state vector*, $\mathbf{x}$, *and a core position vector*, $\hat{\mathbf{s}}\in X$, *in an autonomous*, *rule-continuous*, *standardized*, *recurrent fuzzy system satisfies the equation* $\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{x}-\hat{\mathbf{s}}\|$, *then the dimension of the elementary hypersquare of* $\mathbf{f}(\mathbf{x})$ *is less than or equals the dimension of the elementary hypersquare of* $\mathbf{x}$. *If the dimensions are equal*, *then all vertices of the elementary hypersquare of* $\mathbf{x}$ *are mapped onto vertices of the elementary hypersquare of* $\mathbf{f}(\mathbf{x})$.

Proof. The set $A=\{\mathbf{j}|\prod_i \mu_{j_i}^{x_i}(x_i)\neq 0\}$ contains the indices of all vertices of the elementary hypersquare of $\mathbf{x}$. The set $S=\{\mathbf{s}_\mathbf{j}^\mathbf{x}|\mathbf{j}\in A\}$ contains all vertices of the elementary hypersquare of $\mathbf{x}$, whose dimension is denoted as d.

Consider the two core position vectors, $\mathbf{M}$ and $\mathbf{m}$, from S satisfying $\|\mathbf{M}-\hat{\mathbf{s}}\| = \max_{\mathbf{j}\in A}\|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\|$ and $\|\mathbf{m}-\hat{\mathbf{s}}\| = \min_{\mathbf{j}\in A}\|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\|$, respectively. $\mathbf{M}$ and $\mathbf{m}$ are two diagonally opposed vertices of the d-dimensional hypersquare, because $\mathbf{m}$ takes on those core positions in each component m_i, that are as near as possible to those of $\hat{\mathbf{s}}$ and $\mathbf{M}$ takes on those core positions in each component M_i, that are as far away as possible from those of $\hat{\mathbf{s}}$.

Choose an arbitrary vertex, $\mathbf{z}\in S$, of the hypersquare. Starting from $\mathbf{m}$ by substituting those components one by one by those of $\mathbf{M}$, in which $\mathbf{m}$ and $\mathbf{M}$ differ, one has done exactly d substitutions when arriving at $\mathbf{M}$. The order of those substitution can be chosen such that the vector $\mathbf{z}$ is reached as intermediate vector. The number of components, in which the vectors $\mathbf{z}$ and $\mathbf{m}$ and the vectors $\mathbf{z}$ and $\mathbf{M}$ differ, add up to d. This is also true for the distances between $\mathbf{z}$ and $\mathbf{m}$ and the distances between $\mathbf{z}$ and $\mathbf{M}$, as each of their components differs by 1 at most. Together with the nonexpansion property of $\mathbf{f}$ and the triangular inequality it follows:

$$d = \|\mathbf{M}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{m}\| \geqslant \|\mathbf{f}(\mathbf{M})-\mathbf{f}(\mathbf{z})\| + \|\mathbf{f}(\mathbf{z})-\mathbf{f}(\mathbf{m})\| \geqslant \|\mathbf{f}(\mathbf{M})-\mathbf{f}(\mathbf{m})\|. \tag{A.19}$$

By employing the triangular inequality and Lemma 4 we get that

$$\|\mathbf{f}(\mathbf{M})-\mathbf{f}(\mathbf{m})\| \geqslant \|\mathbf{f}(\mathbf{M})-\mathbf{f}(\hat{\mathbf{s}})\| - \|\mathbf{f}(\mathbf{m})-\mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{M}-\hat{\mathbf{s}}\| - \|\mathbf{m}-\hat{\mathbf{s}}\|. \tag{A.20}$$

Since $\mathbf{M}$ and $\mathbf{m}$ are two diagonally opposed vertices of the d-dimensional hypersquare, we get:

$$\|\mathbf{M}-\hat{\mathbf{s}}\| - \|\mathbf{m}-\hat{\mathbf{s}}\| = \max_{\mathbf{j}\in A}\|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\| - \min_{\mathbf{j}\in A}\|\mathbf{s}_\mathbf{j}^\mathbf{x}-\hat{\mathbf{s}}\| = d. \tag{A.21}$$

Thus all terms in the chain inequality given by Eq. (A.19)–(A.21) are equal and we have particularly

$$\|\mathbf{f}(\mathbf{M})-\mathbf{f}(\mathbf{z})\| + \|\mathbf{f}(\mathbf{z})-\mathbf{f}(\mathbf{m})\| = \|\mathbf{f}(\mathbf{M})-\mathbf{f}(\mathbf{m})\| = d. \tag{A.22}$$

Inserting the definition of the canonical vector norm in Eq. (A.22), we obtain

$$\sum_l \frac{1}{\Delta s_l}|f_l(\mathbf{M})-f_l(\mathbf{z})| + \sum_l \frac{1}{\Delta s_l}|f_l(\mathbf{z})-f_l(\mathbf{m})| = \sum_l \frac{1}{\Delta s_l}|f_l(\mathbf{M})-f_l(\mathbf{m})|. \tag{A.23}$$

Employing the triangular inequality, we have anyway

$$|f_l(\mathbf{M})-f_l(\mathbf{z})| + |f_l(\mathbf{z})-f_l(\mathbf{m})| \geqslant |f_l(\mathbf{M})-f_l(\mathbf{m})|. \tag{A.24}$$

In order to satisfy Eq. (A.23),

$$|f_l(\mathbf{M})-f_l(\mathbf{z})| + |f_l(\mathbf{z})-f_l(\mathbf{m})| = |f_l(\mathbf{M})-f_l(\mathbf{m})| \tag{A.25}$$

must hold for each index l. Therefore, we get for any indices l either

$$f_l(\mathbf{m}) \leqslant f_l(\mathbf{z}) \leqslant f_l(\mathbf{M}) \quad \text{or} \quad f_l(\mathbf{m}) \geqslant f_l(\mathbf{z}) \geqslant f_l(\mathbf{M}). \tag{A.26}$$

The vectors $\mathbf{f}(\mathbf{z})$ may only differ in those components, in which those of $\mathbf{f}(\mathbf{M})$ and $\mathbf{f}(\mathbf{m})$ differ and therefore in no more than d components according to Eq. (A.22). Since $\mathbf{f}(\mathbf{x})$ is computed as a convex combination, i.e., a linear combination with premultipliers that are taken from the interval $[0,1]$ and add up to 1, of those vectors $\mathbf{f}(\mathbf{z})$, the elementary hypersquare of $\mathbf{f}(\mathbf{x})$ is d-dimensional at most. Thus the first part of the lemma is proven.

If the elementary hypersquare of $\mathbf{f}(\mathbf{x})$ is also d-dimensional, then $\mathbf{f}(\mathbf{M})$ and $\mathbf{f}(\mathbf{m})$ have to differ in at least d components. On the other hand, they may differ in no more than d components, as we have $\|\mathbf{f}(\mathbf{M}) - \mathbf{f}(\mathbf{m})\| = d$ according to Eq. (A.22). Following these two arguments, we obtain that they differ exactly in d components and differ by exactly one unit each in order to satisfy the limitation in their distance given in Eq. (A.22). They form two diagonally opposed vertices of a d-dimensional elementary hypersquare, H.

According to Eq. (A.26) the vectors $\mathbf{f}(\mathbf{z})$ for all $\mathbf{z} \in S$ are vertices of the elementary hypersquare H. As $\mathbf{f}(\mathbf{x})$ is computed by a convex combination of those vectors and is located in a d-dimensional elementary hypersquare, it is a vector of H. Thus the second part is proven. □

Lemma 6. *Each accumulation point*, $\mathbf{x}$, *of a trajectory in the state space*, X, *of an autonomous, terminated, rule-continuous, standardized, recurrent fuzzy system is a fixed point and its elementary hypersquare completely consists of equilibrium points.*

Proof. Assume $\mathbf{x} \in X$ is an accumulation point of a time series in X. For any $\varepsilon > 0$ we can find a vector, $\mathbf{a}$, in this time series such that for certain, arbitrarily high, indices, k_0, both $\mathbf{a}$ and $\mathbf{f}^{k_0}(\mathbf{a})$ have a distance smaller than $\varepsilon/2$ to the vector $\mathbf{x}$. Since the function, $\mathbf{f}$, is nonexpansive, we then have that

$$\|\mathbf{f}^{k_0}(\mathbf{x}) - \mathbf{x}\| \leqslant \|\mathbf{f}^{k_0}(\mathbf{x}) - \mathbf{f}^{k_0}(\mathbf{a})\| + \|\mathbf{f}^{k_0}(\mathbf{a}) - \mathbf{x}\| \leqslant \|\mathbf{x} - \mathbf{a}\| + \|\mathbf{f}^{k_0}(\mathbf{a}) - \mathbf{x}\| < \varepsilon. \tag{A.27}$$

and the new time series $\mathbf{f}^k(\mathbf{x})$ approaches $\mathbf{x}$ arbitrarily closely.

Since the recurrent fuzzy system is terminated, it has an end node, whose core position vector, $\hat{\mathbf{s}}$, in the state space, X, is a fixed point. Then we have that $\|\mathbf{f}^k(\mathbf{x}) - \hat{\mathbf{s}}\| = \|\mathbf{f}^k(\mathbf{x}) - \mathbf{f}^k(\hat{\mathbf{s}})\| \leqslant \|\mathbf{x} - \hat{\mathbf{s}}\|$. The strict inequality $\|\mathbf{f}^k(\mathbf{x}) - \hat{\mathbf{s}}\| = \|\mathbf{f}^k(\mathbf{x}) - \mathbf{f}^k(\hat{\mathbf{s}})\| < \|\mathbf{x} - \hat{\mathbf{s}}\|$ for a $k \in \mathbb{N}$ would lead to $\varepsilon < \|\mathbf{x} - \hat{\mathbf{s}}\| - \|\mathbf{f}^k(\mathbf{x}) - \hat{\mathbf{s}}\| \leqslant \|\mathbf{x} - \hat{\mathbf{s}}\| - \|\mathbf{f}^{k_0}(\mathbf{x}) - \hat{\mathbf{s}}\| \leqslant \|\mathbf{f}^{k_0}(\mathbf{x}) - \mathbf{x}\|$ for an $\varepsilon > 0$ and for all $k_0 > k \in \mathbb{N}$ in contradiction to Eq. (A.27).

Thus, we have $\|\mathbf{f}^k(\mathbf{x}) - \hat{\mathbf{s}}\| = \|\mathbf{f}^k(\mathbf{x}) - \mathbf{f}^k(\hat{\mathbf{s}})\| = \|\mathbf{x} - \hat{\mathbf{s}}\|$ for all k. Particularly for $k = 1$, we obtain the equation $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{x} - \hat{\mathbf{s}}\|$ and therefore the premises for Lemmas 4 and 5 are fulfilled. According to Lemma 5, the dimension of the elementary hypersquares of the vectors $\mathbf{f}^k(\mathbf{x})$ does not increase with increasing indices k. The dimensions can not decrease, since the time series $\mathbf{f}^k(\mathbf{x})$ approaches $\mathbf{x}$ arbitrarily closely. Thus Lemmas 4 and 5 are applicable in the extended version.

Analogously to the proof of Lemma 5, we introduce sets A, S and the core position vectors $\mathbf{M}$ and $\mathbf{m}$: the set $A = \{\mathbf{j} | \prod_i \mu_{j_i}^{x_i}(x_i) \neq 0\}$ contains the indices of all vertices of the elementary hypersquare of $\mathbf{x}$. The set $S = \{\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} | \mathbf{j} \in A\}$ consists of vertices of this hypersquare. The core position vectors $\mathbf{M}$ and $\mathbf{m}$ from S satisfy $\|\mathbf{M} - \hat{\mathbf{s}}\| = \max_{\mathbf{j} \in A} \|\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} - \hat{\mathbf{s}}\|$ and $\|\mathbf{m} - \hat{\mathbf{s}}\| = \min_{\mathbf{j} \in A} \|\mathbf{s}_{\mathbf{j}}^{\mathbf{x}} - \hat{\mathbf{s}}\|$, respectively.

Now consider the time series $\mathbf{f}^k(\mathbf{M})$. As the recurrent fuzzy system is terminated, and $\mathbf{M}$ is a core position vector, the series reaches a final core position vector, $\mathbf{M}_\infty$, of an end node after a finite number of steps and remains there. In addition, $\mathbf{f}^k(\mathbf{M})$ is a vertex of the elementary hypersquare of $\mathbf{f}^k(\mathbf{x})$ (according to Lemma 5) that has the greatest distance to $\hat{\mathbf{s}}$ (according to Lemma 4) for all $k \in \mathbb{N}$. Since the series $\mathbf{f}^k(\mathbf{x})$ approaches $\mathbf{x}$ arbitrarily closely, the series $\mathbf{f}^k(\mathbf{M})$ contains the core position vector $\mathbf{M}$ again and again. Thus $\mathbf{M}$ is an accumulation point of the series $\mathbf{f}^k(\mathbf{M})$ and therefore is identical to the limit $\mathbf{M}_\infty$. Thus $\mathbf{M}$ is a fixed point. Analogously it follows that $\mathbf{m}$ is a fixed point as well. Thus the elementary hypersquares of the vectors $\mathbf{f}^k(\mathbf{x})$ are identical for all k. According to Lemma 5 the function $\mathbf{f}$ maps the set S into itself.

Assume there is a core position vector in S that is not a fixed point. Then there is a core position vector, $\mathbf{z}$, in S, whose image $\mathbf{f}(\mathbf{z}) \neq \mathbf{z}$ is a core position vector of an end node, since the transition function maps the set S into itself and since the recurrent fuzzy system is terminated. As the fixed point $\hat{\mathbf{s}}$ can be chosen arbitrarily, we can choose $\hat{\mathbf{s}} = \mathbf{f}(\mathbf{z})$. By employing Lemma 4, we obtain the following contradiction: $0 = \|\mathbf{f}(\mathbf{z}) - \mathbf{f}(\hat{\mathbf{s}})\| = \|\mathbf{z} - \hat{\mathbf{s}}\| \neq 0$. Thus all core position vectors from S, which are the vertices of the elementary hypersquare of $\mathbf{x}$, are fixed points, i.e., $\mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}) = \mathbf{s}_\mathbf{j}^\mathbf{x}$ for all $\mathbf{j} \in A$.

In combination with Lemmas 2 and 3, we obtain for an arbitrary vector $\tilde{\mathbf{x}}$ taken from the elementary hypersquare of $\mathbf{x}$:

$$\mathbf{f}(\tilde{\mathbf{x}}) = \sum_{\mathbf{j} \in A} \mathbf{f}(\mathbf{s}_\mathbf{j}^\mathbf{x}) \prod_i \mu_{j_i}^{x_i}(\tilde{x}_i) = \sum_{\mathbf{j} \in A} \mathbf{s}_\mathbf{j}^\mathbf{x} \prod_i \mu_{j_i}^{x_i}(\tilde{x}_i) = \tilde{\mathbf{x}} \tag{A.28}$$

and thus the elementary hypersquare of $\mathbf{x}$ completely consists of fixed points. In particular, $\mathbf{x}$ is a fixed point, as well. □

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